Weighted Average-convexity and Cooperative Games

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Outline

We generalize the notion of average-convexity to weighted average-convexity.

- We extend a result about the Shapley value and the core to the weighted Shapley value.
- We investigate inheritance of weighted average-convexity for communication TU-games.
  - Necessary conditions.
  - Extension of some known conditions for inheritance of average convexity.

Definitions

Set of players $N = \{1, 2, \ldots, n\}$.

Cooperative TU game (N, v) :
$v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. Coalition $S \subseteq N \to$ worth $v(S)$.

An *allocation* is a vector $x \in \mathbb{R}^N$ representing the respective payoff of each player. It is *efficient* if

$$\sum_{i \in N} x_i = v(N).$$

and *individually rational* if

$$\forall i \in N,\ x_i \geq v(\{i\}).$$

Shapley value

The Shapley value of a cooperative game (N, v) is an allocation vector $\Phi \in \mathbb{R}^N$ assigning to each player $i \in N$:

$$\Phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!}(v(S \cup \{i\}) - v(S)).$$

Decomposition into unanimity games

[Shapley, 1953b] : Every cooperative game (N, v) can be written as a unique linear combination of unanimity games,

$$v = \sum_{S \subseteq N} \lambda_S(v) u_S,$$

where $\lambda_\emptyset(v) = 0$, and $\forall S \neq \emptyset$ the coefficients $\lambda_S(v)$ are given by

$$\lambda_S(v) = \sum_{T \subseteq S} (-1)^{s-t} v(T).$$

Shapley value

Definition

The **Shapley value** is the unique function from the set of TU-games to payoff allocations such that

1. It is linear,
2. The allocation of the unanimity game u_S is for all $i \in N$,

$$x_i = \begin{cases} \frac{1}{s}, & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

In terms of the unanimity coefficients the Shapley value is given by

$$\Phi_i(v) = \sum_{S \subseteq N:\, i \in S} \frac{1}{s} \lambda_S(v),$$

for all $i \in N$.

Core

Definition

The *core* is the set of payoff allocations satisfying efficiency and coalitional rationality. Formally,

$$\mathcal{C}(v) = \left\{ x \in \mathbb{R}^N, \ \sum_{i \in N} x_i = v(N), \ \sum_{i \in S} x_i \geq v(S), \forall S \subset N \right\}.$$

Condition ensuring that the Shapley value lies in the core ?

- Convexity

Convexity

Definition

The game (N, v) is *convex* if for every $S, T \subseteq N$

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T),$$

or equivalently if for all $i \in N$ and for all $S \subseteq T \subseteq N \setminus \{i\}$

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

$\rightarrow$ Tendency to join the largest coalitions.

Convexity ensures good properties, in particular

- Non-emptiness of the core.
- Shapley value belongs to the core.

A weaker sufficient condition ?

Average convexity

Definition

The game (N, v) is *average convex* if for every $S \subset T \subseteq N$,

$$\sum_{i \in S} (v(S) - v(S \setminus \{i\})) \leq \sum_{i \in S} (v(T) - v(T \setminus \{i\})) .$$

Proposition ([Iñarra and Usategui, 1993])

If the game is average convex then the Shapley value is in the core.

Weighted Shapley value

The Shapley value has been extended in [Shapley, 1953a] and in [Kalai and Samet, 1987] to weighted Shapley value.

Weights on the players : $i \in N \rightarrow$ weight $\omega_i \in \mathbb{R}_+^N$

Priorities on the players : $i \in N \rightarrow$ priority $p(i) \in \{1, 2, \ldots, m\}$ with $m \leq n$.
$\rightarrow$ N can be partitionned into m subsets $(N_1, ..., N_m)$ corresponding to the m levels of priority.

Weight relative to a coalition $S \subseteq N$: player $i \in S$ gets weight ω_i^S with

$$\omega_i^S = \begin{cases} \omega_i & \text{if } i \text{ has highest priority in } S \\ 0 & \text{otherwise} \end{cases}$$

Weighted Shapley Value - Weight system

Definition

A **weight system** is a pair (ω, Σ) where $\omega \in \mathbb{R}^N_{++}$ and $\Sigma = (N_1, ..., N_m)$ is an ordered partition of N.

Players in N_k have priority k.

Given a set S, the priority $p(S)$ of S is the largest $k \in \{1, \cdots, m\}$ such that $N_k \cap S \neq \emptyset$.

$\overline{S}$:= set of players in S with highest priority, i.e.,

$$\overline{S} = \{i \in S, p(i) = p(S)\}.$$

If $m = 1$ then $\Sigma = N$.

Weighted Shapley Value

Definition

The **weighted Shapley value** with weight system (ω, Σ) is the unique function from the set of TU-games to allocation such that

1. it is linear,
2. the allocation of the unanimity game u_S is defined as follows : for all $i \in N$,

$$x_i = \frac{\omega_i^S}{\sum_{i \in S} \omega_i^S} = \begin{cases} \frac{\omega_i}{\sum_{i \in \overline{S}} \omega_i}, & \text{if } i \in \overline{S}, \\ 0 & \text{otherwise.} \end{cases}$$

- agents in $S - \overline{S}$ are contributing to obtain a positive payoff but they have low priority, hence they obtain 0,
- agents in $\overline{S}$ are contributing to obtain a positive payoff and have highest priority in S, hence they share the total value of 1.

Weighted Shapley Value

Using the decomposition of a game into unanimity games, the (ω, Σ)-weighted Shapley value Φ^ω of a game (N, v) is defined for all $i \in N$ by

$$\Phi_i^\omega(v) = \sum_{S \subseteq N \colon i \in \overline{S}} \frac{\omega_i}{\overline{\omega}^S} \lambda_S(v).$$

- If $\Sigma = \{N\}$ and if all weights are equal, then the (ω, Σ)-weighted Shapley value corresponds to the Shapley value.

Weighted average convexity

We introduce the notion of weighted average convexity.

Definition

Let (ω, Σ) be a weight system. The game (N, v) is (ω, Σ)-**convex** if for every $S \subset T \subseteq N$,

$$\sum_{i \in S} \overline{\omega_i}^T \left(v(S) - v(S \setminus \{i\}) \right) \leq \sum_{i \in S} \overline{\omega_i}^T \left(v(T) - v(T \setminus \{i\}) \right).$$

- It is sufficient to consider subsets such that $p(S) = p(T)$.
- If $\Sigma = \{N\}$ and if all weights are equal, then (ω, Σ)-convexity corresponds to average-convexity.
- If a game is convex then it is (ω, Σ)-convex for any weight system (ω, Σ).

We get the following result

Theorem

Let (ω, Σ) be a weight system. If the game is (ω, Σ)-convex then its (ω, Σ)-weighted Shapley value is in the core.

- We establish a recurrence formula for the weighted Shapley value. For any $\emptyset \neq T \subseteq N$, let v^T be the subgame of v induced by T. *i.e.*, $v^T(S) = v(S)$ for any $S \subseteq T$. We have

$$\Phi^{\omega}_{iT} = \frac{\overline{\omega}_i^T}{\overline{\omega}^T}(v(T) - v(T \setminus \{i\})) + \sum_{j \in T \setminus \{i\}} \frac{\overline{\omega}_j^T}{\overline{\omega}^T} \Phi^{\omega}_{iT \setminus \{j\}},$$

  for all $i \in T$.
- Then we can prove the theorem by recurrence on the number of players.

1. Weighted average convexity and Shapley value
2. Inheritance of weighted average convexity

Inheritance of properties

Coalition $\rightarrow$ partition into (sub)coalitions $\rightarrow$ Restricted game

1. Conditions insuring inheritance of convexity
2. Conditions for inheritance of average convexity
3. Conditions for inheritance of weighted average convexity

Myerson's restricted game

- Results for 1 and 2 have been established by [van den Nouweland and Borm, 1991] and [Slikker, 1998] respectively.
- We investigate 3 : inheritance of weighted average convexity.

Myerson's restricted game

Cooperative game (N, v) and graph $G = (N, E)$.

nodes	$\leftrightarrow$	players
edge $e = \{i, j\}$	$\leftrightarrow$	players i and j can communicate directly

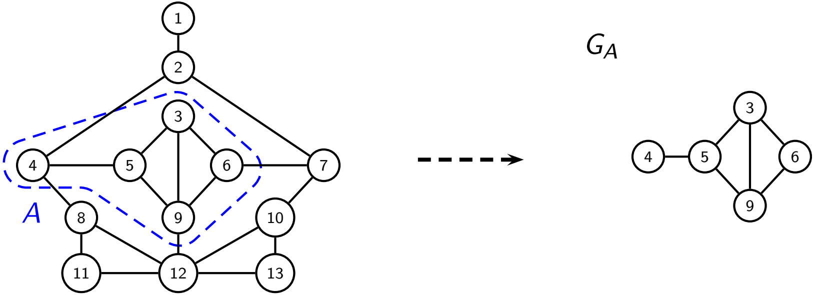
For every coalition $A \subseteq N$, let $\mathcal{P}_c(A)$ be the set of connected components of $G_A = (A, E(A))$.

Myerson defined the graph-restricted game (N, v^G) by :

$$v^G(A) = \sum_{F \in \mathcal{P}_c(A)} v(F), \ \forall A \subseteq N.$$

- Players have to be connected to cooperate.
- Connectedness is sufficient.

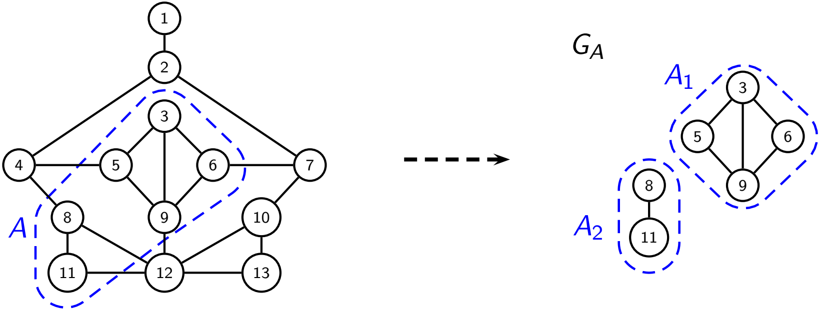
If G_A is connected



$$v^G(A) = v(A).$$

If G_A is non-connected, let $\{A_1, A_2, \ldots, A_k\}$ be the partition of A, then

$$v^G(A) = \sum_{j=1}^{k} v(A_j).$$



$$v^G(A) = v(A_1) + v(A_2).$$

Inheritance of convexity

- Conditions on the underlying graph

Definition

A cycle $C = \{v_1, e_1, v_2, e_2, \dots, v_m, e_m, v_1\}$ is **complete** (resp. non-complete) if the subset $\{v_1, v_2, \dots v_m\} \subseteq N$ of vertices of C induces a complete (resp. non-complete) subgraph.

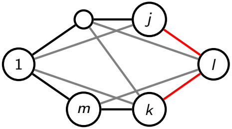


FIGURE – Non-complete cycle C, $\{j, k\} \notin E$.

Definition

A graph $G = (N, E)$ is **cycle-complete** if any cycle C in G is complete.

Inheritance of convexity

Forbidden subgraphs :

- Non-complete cycle

Theorem (van den Nouweland and Borm 1991)

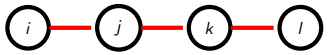
Let $G = (N, E)$ be a connected graph. The following properties are equivalent.

1. *G preserves convexity*
2. *G is cycle-complete.*

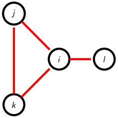
Inheritance of average-convexity

Forbidden subgraphs :

- Non-complete cycle
- 4-path
- 3-pan



(a) 4-path.



(b) 3-pan.

Theorem (Slikker)

Let $G = (N, E)$ be a connected graph. The following properties are equivalent.

1. *G preserves average-convexity.*
2. 1. *G is cycle-complete.*
   2. *There is no restricted subgraph that is a 4-path or a 3-pan.*
3. *G is a complete graph or a star.*

Inheritance of weighted average convexity

First Case : All players have the same priority, $\Sigma = \{N\}$.

- Players can have different weights.

We get the same characterization as Slikker with average convexity.

Theorem

Let $G = (N, E)$ be a connected graph and let (ω, Σ) be a weight system with $\Sigma = \{N\}$. The following properties are equivalent.

1. *G preserves (ω, Σ)-convexity.*
2. 1. *G is cycle-complete.*
   2. *There is no restricted subgraph that is a 4-path or a 3-pan.*
3. *G is a complete graph or a star.*

Proof

- Similarly to Slikker we have to prove that G cannot contain any 4-path or 3-pan.
- Counter-examples are more difficult as they have to be valid for arbitrary weights.

Counter-Example (Weighted Non-complete cycle)

Let j and k be neighbors of l^* in C with $\{j, k\} \notin E$. We consider the convex game defined by $v(S) = |S| - 1$, $\forall S \subseteq N$, $S \neq \emptyset$.

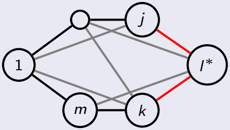


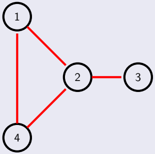
FIGURE – Non-complete cycle C, $\{j, k\} \notin E$.

Taking $S = \{j, l^*, k\}$ and $T = V(C)$, we get

$$\sum_{i \in S} \omega_i(v^G(S) - v^G(S \setminus \{i\})) = \omega_j + 2\omega_{l^*} + \omega_k > \omega_j + \omega_{l^*} + \omega_k = \sum_{i \in S} \omega_i(v^G(T) - v^G(T \setminus \{i\})).$$

This contradicts (ω, Σ)-convexity of (N, v^G).

Counter-Example (3-pan)



FIGURE

$X = 1 + \frac{\omega_3}{\omega_4}$,

$Y = 1 + \frac{\omega_1}{\omega_4}$,

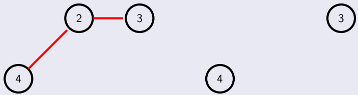
$Z = X + Y + 1 + \frac{\omega_1}{\omega_2 + \omega_3 + \omega_4} X$,

$\Theta = Z + X - 1$.

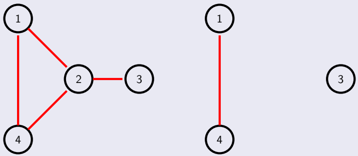
$$v(S) = \begin{cases} 0 \text{ if } |S| \in \{0,1,2\} \text{ and } S \neq \{1,4\},\{3,4\}, \\ 0 \text{ if } S = \{1,2,3\}, \\ X \text{ if } S = \{1,4\} \text{ or } \{1,2,4\}, \\ Y \text{ if } S = \{3,4\}, \\ X + Y - 1 \text{ if } S = \{1,3,4\}, \\ Z \text{ if } S = \{2,3,4\}, \\ \Theta \text{ if } S = N. \end{cases}$$

v is weighted average convex. But v^G is not. We get a contradiction with $S = \{2,3,4\} \subset T = \{1,2,3,4\}$.

Counter-Example (3-pan)



(a) G_S (b) $G_S \setminus \{2\}$



(a) G_T (b) $G_T \setminus \{2\}$

$$v(S) = \begin{cases} 0 \text{ if } |S| \in \{0,1,2\} \text{ and } S \neq \{1,4\},\{3,4\}, \\ 0 \text{ if } S = \{1,2,3\}, \\ X \text{ if } S = \{1,4\} \text{ or } \{1,2,4\}, \\ Y \text{ if } S = \{3,4\}, \\ X + Y - 1 \text{ if } S = \{1,3,4\}, \\ Z \text{ if } S = \{2,3,4\}, \\ \Theta \text{ if } S = N. \end{cases} \quad v^G(S) = \begin{cases} 0 \text{ if } |S| \in \{0,1,2\} \text{ and } S \neq \{1,4\},\{3,4\}, \\ 0 \text{ if } S = \{1,2,3\}, \\ X \text{ if } S = \{1,4\} \text{ or } \{1,2,4\}, \\ 0 \text{ if } S = \{3,4\}, \\ X \text{ if } S = \{1,3,4\}, \\ Z \text{ if } S = \{2,3,4\}, \\ \Theta \text{ if } S = N \end{cases}$$

Remark

The previous counter-example is also valid for the 4-path.

Inheritance of weighted average convexity

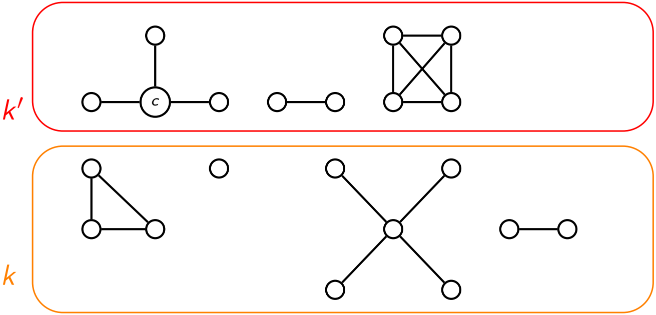
Second Case : **Players with different priorities, $\Sigma \neq \{N\}$.**

- Using the preceding results, the situation for players in a given priority layer can be easily established.

Proposition

If a graph (N, E) preserves the (ω, Σ)-convexity, given a priority k, the set of players of priority k corresponds to a collection of disconnected star/complete subgraphs.

Inside priority layers



Inheritance of weighted average convexity

- Links between layers ?

The previous counter-examples have to be refined and supplementary conditions are required.

Inheritance of weighted average convexity

We get a similar counterexample for non-complete cycles but only if $p(l^*) = p(V(C))$.

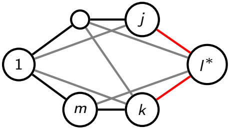


FIGURE – Non-complete cycle C, $\{j,k\} \notin E$.

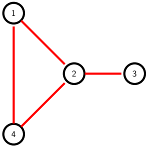
Taking $S = \{j, l^*, k\}$ and $T = V(C)$, we get

$$\sum_{i \in S} \omega_i^T (v^G(S) - v^G(S \setminus \{i\})) = \omega_j^T + 2\omega_{l^*}^T + \omega_k^T$$

$$> \omega_j^T + \omega_{l^*}^T + \omega_k^T =$$

$$\sum_{i \in S} \omega_i^T (v^G(T) - v^G(T \setminus \{i\})).$$

Inheritance of weighted average convexity



FIGURE

The previous example on the 3-pan is now valid only if

$$p(2) = p(3) = p(4) \geq p(1),$$

or

$$p(2) > p(4) \geq \max(p(1), p(3)).$$

- We established 2 supplementary counter-examples for other priority distributions.
- We get a very precise outline if the communication graph is cycle-free.

Inheritance of weighted average convexity

Lemma

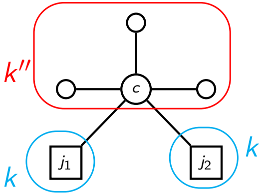
Let $G = (N, E)$ *be a* **cycle-free** *graph preserving* (ω, Σ)*-convexity. Let* $k \leq k' < k''$ *be priority levels. Let* C_1 *(resp.* C_2*) be a component of* G_k *(resp.* $G_{k'}$*) linked to a component* C *of* $G_{k''}$*. Then the following statements are satisfied :*

1. C *and* C_1 *are stars (possibly of size* 1 *or* 2*).*
2. C_2 *is a singleton.*
3. C_2 *is linked to* C *only at its center* c*.*
4. C_1 *is linked to* C *only at its center* c *by a unique edge.*
5. C_2 *cannot be linked to connected components of a lower layer.*

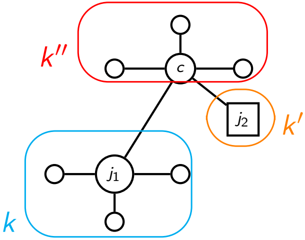
Moreover, if $k = k'$*, then*

1. C_1 *is a singleton.*
2. C_1 *cannot be linked to connected components of a lower layer.*

Inheritance of weighted average convexity



(a) If $k = k'$ then C_1 and C_2 are singletons.



(b) $k < k'$, C_2 is a singleton.

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