Weighted Average-convexity and Cooperative Games

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We generalize the notion of average-convexity to weighted average-convexity.

- We extend a result about the Shapley value and the core to the weighted Shapley value.

- We investigate inheritance of weighted average-convexity for communication TU-games.
  - Necessary conditions.
  - Extension of some known conditions for inheritance of average convexity.
1. Weighted average convexity and Shapley value

2. Inheritance of weighted average convexity
Definitions

Set of players $N = \{1, 2, \ldots, n\}$.

Cooperative TU game $(N, v)$:
$v : 2^N \rightarrow \mathbb{R}$, $v(\emptyset) = 0$. Coalition $S \subseteq N \rightarrow$ worth $v(S)$.

An allocation is a vector $x \in \mathbb{R}^N$ representing the respective payoff of each player. It is efficient if
$$\sum_{i \in N} x_i = v(N).$$

and individually rational if
$$\forall i \in N, \ x_i \geq v(\{i\}).$$
The Shapley value of a cooperative game \((N, \nu)\) is an allocation vector \(\Phi \in \mathbb{R}^N\) assigning to each player \(i \in N\):

\[
\Phi_i(\nu) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} (\nu(S \cup \{i\}) - \nu(S)).
\]
[Shapley, 1953b] : Every cooperative game \((N, \nu)\) can be written as a unique linear combination of unanimity games,

\[
\nu = \sum_{S \subseteq N} \lambda_S(\nu) u_S,
\]

where \(\lambda_\emptyset(\nu) = 0\), and \(\forall S \neq \emptyset\) the coefficients \(\lambda_S(\nu)\) are given by

\[
\lambda_S(\nu) = \sum_{T \subseteq S} (-1)^{s-t} \nu(T).
\]
The **Shapley value** is the unique function from the set of \(TU\)-games to payoff allocations such that

1. It is linear,
2. The allocation of the unanimity game \(u_S\) is for all \(i \in N\),

\[
x_i = \begin{cases} 
\frac{1}{S}, & \text{if } i \in S, \\
0, & \text{otherwise.}
\end{cases}
\]

In terms of the unanimity coefficients the Shapley value is given by

\[
\Phi_i(v) = \sum_{S \subseteq N : i \in S} \frac{1}{S} \lambda_S(v),
\]

for all \(i \in N\).
Definition

The core is the set of payoff allocations satisfying efficiency and coalitional rationality. Formally,

\[ C(v) = \left\{ x \in \mathbb{R}^N, \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S), \forall S \subset N \right\}. \]

Condition ensuring that the Shapley value lies in the core?

- Convexity
The game \((N, \nu)\) is convex if for every \(S, T \subseteq N\)

\[
\nu(S) + \nu(T) \leq \nu(S \cup T) + \nu(S \cap T),
\]

or equivalently if for all \(i \in N\) and for all \(S \subseteq T \subseteq N \setminus \{i\}\)

\[
\nu(S \cup \{i\}) - \nu(S) \leq \nu(T \cup \{i\}) - \nu(T).
\]

→ Tendency to join the largest coalitions.

Convexity ensures good properties, in particular

- Non-emptiness of the core.
- Shapley value belongs to the core.

A weaker sufficient condition?
Definition

The game \((N, \nu)\) is average convex if for every \(S \subseteq T \subseteq N\),

\[
\sum_{i \in S} (\nu(S) - \nu(S \setminus \{i\})) \leq \sum_{i \in S} (\nu(T) - \nu(T \setminus \{i\})).
\]

Proposition ([Iñarra and Usategui, 1993])

If the game is average convex then the Shapley value is in the core.
The Shapley value has been extended in [Shapley, 1953a] and in [Kalai and Samet, 1987] to weighted Shapley value.

**Weights on the players:** \( i \in N \rightarrow \text{weight} \ \omega_i \in \mathbb{R}_+^N \)

**Priorities on the players:** \( i \in N \rightarrow \text{priority} \ p(i) \in \{1, 2, \ldots, m\} \) with \( m \leq n \).

\( N \) can be partitionned into \( m \) subsets \((N_1, \ldots, N_m)\) corresponding to the \( m \) levels of priority.

**Weight relative to a coalition** \( S \subseteq N \): player \( i \in S \) gets weight \( \omega_i^S \) with

\[
\omega_i^S = \begin{cases} 
\omega_i & \text{if } i \text{ has highest priority in } S \\
0 & \text{otherwise}
\end{cases}
\]
A weight system is a pair \((\omega, \Sigma)\) where \(\omega \in \mathbb{R}^{N}_{++}\) and \(\Sigma = (N_1, \ldots, N_m)\) is an ordered partition of \(N\).

Players in \(N_k\) have priority \(k\).

Given a set \(S\), the priority \(p(S)\) of \(S\) is the largest \(k \in \{1, \cdots, m\}\) such that \(N_k \cap S \neq \emptyset\).

\(\overline{S} := \text{set of players in } S \text{ with highest priority, i.e.,}\)

\[\overline{S} = \{ i \in S, p(i) = p(S) \}.\]

If \(m = 1\) then \(\Sigma = N\).
Weighted Shapley Value

Definition

The **weighted Shapley value** with weight system \((\omega, \Sigma)\) is the unique function from the set of \(TU\)-games to allocation such that

1. it is linear,

2. the allocation of the unanimity game \(u_S\) is defined as follows: for all \(i \in N\),

\[
x_i = \frac{\omega_i^S}{\sum_{i \in S} \omega_i^S} = \begin{cases} \frac{\omega_i}{\sum_{i \in \overline{S}} \omega_i}, & \text{if } i \in \overline{S}, \\ 0, & \text{otherwise}. \end{cases}
\]

- agents in \(S - \overline{S}\) are contributing to obtain a positive payoff but they have low priority, hence they obtain 0,

- agents in \(\overline{S}\) are contributing to obtain a positive payoff and have highest priority in \(S\), hence they share the total value of 1.
Using the decomposition of a game into unanimity games, the \((\omega, \Sigma)\)-weighted Shapley value \(\Phi^\omega\) of a game \((N, \nu)\) is defined for all \(i \in N\) by

\[
\Phi^\omega_i(\nu) = \sum_{S \subseteq N: i \in \overline{S}} \frac{\omega_i}{\omega_S} \lambda_S(\nu).
\]

- If \(\Sigma = \{N\}\) and if all weights are equal, then the \((\omega, \Sigma)\)-weighted Shapley value corresponds to the Shapley value.
Weighted average convexity

We introduce the notion of weighted average convexity.

**Definition**

Let $(\omega, \Sigma)$ be a weight system. The game $(N, v)$ is $(\omega, \Sigma)$-convex if for every $S \subset T \subseteq N$,

$$\sum_{i \in S} \omega_i T (v(S) - v(S \setminus \{i\})) \leq \sum_{i \in S} \omega_i T (v(T) - v(T \setminus \{i\})).$$

- It is sufficient to consider subsets such that $p(S) = p(T)$.
- If $\Sigma = \{N\}$ and if all weights are equal, then $(\omega, \Sigma)$-convexity corresponds to average-convexity.
- If a game is convex then it is $(\omega, \Sigma)$-convex for any weight system $(\omega, \Sigma)$. 
We get the following result

**Theorem**

Let $(\omega, \Sigma)$ be a weight system. If the game is $(\omega, \Sigma)$-convex then its $(\omega, \Sigma)$-weighted Shapley value is in the core.

- We establish a recurrence formula for the weighted Shapley value. For any $\emptyset \neq T \subseteq N$, let $\nu^T$ be the subgame of $\nu$ induced by $T$. i.e., $\nu^T(S) = \nu(S)$ for any $S \subseteq T$. We have

  \[
  \Phi_{i,T}^\omega = \frac{\omega_i^T}{\omega^T}(\nu(T) - \nu(T \setminus \{i\})) + \sum_{j \in T \setminus \{i\}} \frac{\omega_j^T}{\omega^T} \Phi_{i,T\setminus\{j\}}^\omega,
  \]

  for all $i \in T$.

- Then we can prove the theorem by recurrence on the number of players.
1 Weighted average convexity and Shapley value

2 Inheritance of weighted average convexity
Inheritance of properties

Coalition $\rightarrow$ partition into (sub)coalitions $\rightarrow$ Restricted game

1. Conditions insuring inheritance of convexity
2. Conditions for inheritance of average convexity
3. Conditions for inheritance of weighted average convexity

Myerson’s restricted game

- Results for 1 and 2 have been established by [van den Nouweland and Borm, 1991] and [Slikker, 1998] respectively.
- We investigate 3: inheritance of weighted average convexity.
Myerson’s restricted game

Cooperative game \((N, v)\) and graph \(G = (N, E)\).

- nodes \(\leftrightarrow\) players
- edge \(e = \{i, j\} \leftrightarrow\) players \(i\) and \(j\) can communicate directly

For every coalition \(A \subseteq N\), let \(\mathcal{P}_c(A)\) be the set of connected components of \(G_A = (A, E(A))\).

Myerson defined the graph-restricted game \((N, v^G)\) by:

\[
v^G(A) = \sum_{F \in \mathcal{P}_c(A)} v(F), \quad \forall A \subseteq N.
\]

- Players have to be connected to cooperate.
- Connectedness is sufficient.
Myerson’s restricted game

If $G_A$ is connected

$\nu^G(A) = \nu(A)$. 
If $G_A$ is non-connected, let $\{A_1, A_2, \ldots, A_k\}$ be the partition of $A$, then

$$\nu^G(A) = \sum_{j=1}^{k} \nu(A_j).$$

$$\nu^G(A) = \nu(A_1) + \nu(A_2).$$
Inheritance of convexity

- Conditions on the underlying graph

**Definition**

A cycle $C = \{v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_1\}$ is **complete** (resp. non-complete) if the subset $\{v_1, v_2, \ldots v_m\} \subseteq N$ of vertices of $C$ induces a complete (resp. non-complete) subgraph.

**Figure** – Non-complete cycle $C$, $\{j, k\} \notin E$.

**Definition**

A graph $G = (N, E)$ is **cycle-complete** if any cycle $C$ in $G$ is complete.
Inheritance of convexity

Forbidden subgraphs:
- Non-complete cycle

**Theorem (van den Nouweland and Borm 1991)**

Let $G = (N, E)$ be a connected graph. The following properties are equivalent.

1. $G$ preserves convexity
2. $G$ is cycle-complete.
Forbidden subgraphs:
- Non-complete cycle
- 4-path
- 3-pan

Theorem (Slikker)
Let $G = (N, E)$ be a connected graph. The following properties are equivalent.

1. $G$ preserves average-convexity.
2. $G$ is cycle-complete.
   1. There is no restricted subgraph that is a 4-path or a 3-pan.
3. $G$ is a complete graph or a star.
Inheritance of weighted average convexity

**First Case:** All players have the same priority, $\Sigma = \{N\}$.

- Players can have different weights.

We get the same characterization as Slikker with average convexity.

**Theorem**

Let $G = (N, E)$ be a connected graph and let $(\omega, \Sigma)$ be a weight system with $\Sigma = \{N\}$. The following properties are equivalent.

1. $G$ preserves $(\omega, \Sigma)$-convexity.
2. $G$ is cycle-complete.
   - 1. There is no restricted subgraph that is a 4-path or a 3-pan.
3. $G$ is a complete graph or a star.
Similarly to Slikker we have to prove that $G$ cannot contain any 4-path or 3-pan.

Counter-examples are more difficult as they have to be valid for arbitrary weights.
Counter-Example (Weighted Non-complete cycle)

Let $j$ and $k$ be neighbors of $l^*$ in $C$ with $\{j, k\} \notin E$. We consider the convex game defined by $v(S) = |S| - 1$, $\forall S \subseteq N$, $S \neq \emptyset$.

![Non-complete cycle](image)

**Figure** – Non-complete cycle $C$, $\{j, k\} \notin E$.

Taking $S = \{j, l^*, k\}$ and $T = V(C)$, we get

$$\sum_{i \in S} \omega_i(v^G(S) - v^G(S \setminus \{i\})) = \omega_j + 2\omega_{l^*} + \omega_k > \omega_j + \omega_{l^*} + \omega_k = \sum_{i \in S} \omega_i(v^G(T) - v^G(T \setminus \{i\}))$$.

This contradicts $(\omega, \Sigma)$-convexity of $(N, v^G)$. 
Counter-Example (3-pan)

\[ X = 1 + \frac{\omega_3}{\omega_4}, \]
\[ Y = 1 + \frac{\omega_1}{\omega_4}, \]
\[ Z = X + Y + 1 + \frac{\omega_1}{\omega_2 + \omega_3 + \omega_4} X, \]
\[ \Theta = Z + X - 1. \]

\[ v(S) = \begin{cases} 
0 & \text{if } |S| \in \{0,1,2\} \text{ and } S \neq \{1,4\}, \{3,4\}, \\
0 & \text{if } S = \{1,2,3\}, \\
X & \text{if } S = \{1,4\} \text{ or } \{1,2,4\}, \\
Y & \text{if } S = \{3,4\}, \\
X + Y - 1 & \text{if } S = \{1,3,4\}, \\
Z & \text{if } S = \{2,3,4\}, \\
\Theta & \text{if } S = \emptyset.
\end{cases} \]

\( v \) is weighted average convex.
But \( v^G \) is not.
We get a contradiction with \( S = \{2,3,4\} \subset T = \{1,2,3,4\}. \)
Counter-Example (3-pan)

\[ v(S) = \begin{cases} 
0 & \text{if } |S| \in \{0,1,2\} \text{ and } S \neq \{1,4\}, \{3,4\}, \\
0 & \text{if } S = \{1,2,3\}, \\
X & \text{if } S = \{1,4\} \text{ or } \{1,2,4\}, \\
Y & \text{if } S = \{3,4\}, \\
X + Y - 1 & \text{if } S = \{1,3,4\}, \\
Z & \text{if } S = \{2,3,4\}, \\
\Theta & \text{if } S = N. 
\end{cases} \]

\[ v^G(S) = \begin{cases} 
0 & \text{if } |S| \in \{0,1,2\} \text{ and } S \neq \{1,4\}, \{3,4\}, \\
0 & \text{if } S = \{1,2,3\}, \\
X & \text{if } S = \{1,4\} \text{ or } \{1,2,4\}, \\
0 & \text{if } S = \{3,4\}, \\
X & \text{if } S = \{1,3,4\}, \\
Z & \text{if } S = \{2,3,4\}, \\
\Theta & \text{if } S = N. 
\end{cases} \]
Remark

The previous counter-example is also valid for the 4-path.
**Inheritance of weighted average convexity**

**Second Case:** Players with different priorities, $\Sigma \neq \{N\}$.

Using the preceding results, the situation for players in a given priority layer can be easily established.

**Proposition**

*If a graph $(N, E)$ preserves the $(\omega, \Sigma)$-convexity, given a priority $k$, the set of players of priority $k$ corresponds to a collection of disconnected star/complete subgraphs.*
Inside priority layers

$k'$

$k$

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Inheritance of weighted average convexity

• Links between layers?

The previous counter-examples have to be refined and supplementary conditions are required.
Inheritance of weighted average convexity

We get a similar counterexample for non-complete cycles but only if $p(l^*) = p(V(C))$.

Taking $S = \{j, l^*, k\}$ and $T = V(C)$, we get

$$\sum_{i \in S} \omega_i^T (\nu^G(S) - \nu^G(S \setminus \{i\})) = \omega_j^T + 2\omega_{l^*}^T + \omega_k^T$$

$$> \omega_j^T + \omega_{l^*}^T + \omega_k^T = \sum_{i \in S} \omega_i^T (\nu^G(T) - \nu^G(T \setminus \{i\})).$$

**Figure** – Non-complete cycle $C$, $\{j, k\} \notin E.$
Inheritance of weighted average convexity

The previous example on the 3-pan is now valid only if

\[ p(2) = p(3) = p(4) \geq p(1), \]

or

\[ p(2) > p(4) \geq \max(p(1), p(3)). \]

- We established 2 supplementary counter-examples for other priority distributions.
- We get a very precise outline if the communication graph is cycle-free.
Lemma

Let $G = (N, E)$ be a cycle-free graph preserving $(\omega, \Sigma)$-convexity. Let $k \leq k' < k''$ be priority levels. Let $C_1$ (resp. $C_2$) be a component of $G_k$ (resp. $G_{k'}$) linked to a component $C$ of $G_{k''}$. Then the following statements are satisfied:

1. $C$ and $C_1$ are stars (possibly of size 1 or 2).
2. $C_2$ is a singleton.
3. $C_2$ is linked to $C$ only at its center $c$.
4. $C_1$ is linked to $C$ only at its center $c$ by a unique edge.
5. $C_2$ cannot be linked to connected components of a lower layer.

Moreover, if $k = k'$, then

1. $C_1$ is a singleton.
2. $C_1$ cannot be linked to connected components of a lower layer.
Inheritance of weighted average convexity

(a) If $k = k'$ then $C_1$ and $C_2$ are singletons.

(b) $k < k'$, $C_2$ is a singleton.
The shapley value and average convex games.

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