Axioms for optimal stable rules and fair division rules in a multiple-partners job market

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Outline

The (simple) assignment market

- Two disjoint and finite sets, B and S.
- Each seller $j \in S$ has one indivisible object on sale.
- Each buyer $i \in B$ wants to buy one object and values in $a_{ij} \geq 0$ the object of seller j.
- Each agent has a reservation value:
  $a_{i0} \geq 0$ for each $i \in B$ and $a_{0j} \geq 0$ for each $j \in S$.
- An assignment market is (B, S, a) where
  $a = (a_{ij})_{(i,j) \in (B \cup \{0\}) \times (S \cup \{0'\})}$, with $a_{00} = 0$.
- A matching $\mu \in \mathcal{M}(B, S)$ is a partition of $B \cup S$ in mixed-pair coalitions and singletons.
- Given a market (B, S, a), a matching μ is optimal if
  $\sum_{T \in \mu} a_T \geq \sum_{T \in \mu'} a_T$ for all $\mu' \in \mathcal{M}(B, S)$.

The (simple) assignment market

For the moment we assume $a_{i0} = a_{0j} = 0$...

- A payoff vector $(u, v) \in \mathbb{R}^B \times \mathbb{R}^S$ is feasible for (B, S, a) if there exists $\mu \in \mathcal{M}(B, S)$ such that
  - $u_i + v_j = a_{ij}$ if $(i, j) \in \mu$,
  - $u_i = a_{i0}$ if $i \in B$ unassigned and $v_j = a_{0j}$ if $j \in S$ unassigned.
- Then μ is compatible with (u, v) and $(u, v; \mu)$ is a feasible outcome.
- A feasible outcome is stable for (B, S, a) if
  - $u_i + v_j \geq a_{ij}$ for all $(i, j) \in B \times S$ and
  - $u_i \geq a_{i0}$ for all $i \in B$ and $v_j \geq a_{0j}$ for all $j \in S$.
- It is well-known that if $(u, v; \mu)$ is a stable outcome, then μ is an optimal matching.

The (simple) assignment game

Given an assignment market (B, S, a), the assignment game is $(B \cup S, w_a)$ where, for each $T \subseteq B \cup S$,

$$w_a(T) = \max_{\mu \in \mathcal{M}(B \cap T, S \cap T)} \sum_{(i,j) \in \mu} a_{ij}.$$

- The core of the assignment game is the set of stable payoff vectors (Shapley and Shubik, 1972),
- and it has a lattice structure with an optimal stable payoff vector for each sector: $(\overline{u}(a), \underline{v}(a))$ and $(\underline{u}(a), \overline{v}(a))$.
- Demange (1982) and Leonard (1983): for each $k \in S$,

$$\overline{v}_k(a) = \max_{\mu \in \mathcal{M}(B,S)} \sum_{(i,j) \in \mu} a_{ij} - \max_{\mu \in \mathcal{M}(B, S \setminus \{k\})} \sum_{(i,j) \in \mu} a_{ij}.$$

- The fair division point (Thompson, 1981) is

$$\tau(a) = \frac{1}{2}(\overline{u}(a), \underline{v}(a)) + \frac{1}{2}(\underline{u}(a), \overline{v}(a)).$$

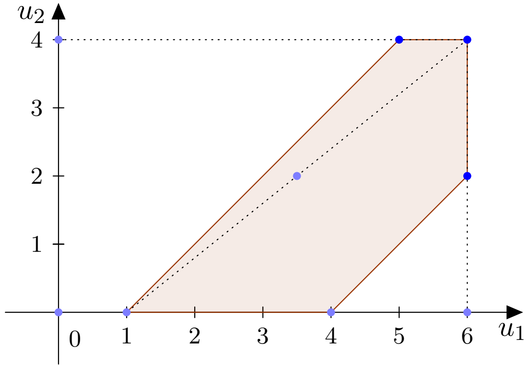
An example

	1'	2'
1	**6**	5
2	2	**4**

.

The core is defined by

$$u_1 + v_1 = 6$$
$$u_1 + v_2 \geq 5$$
$$u_2 + v_1 \geq 2$$
$$u_2 + v_2 = 4$$
$$u_i \geq 0,\ v_j \geq 0.$$



Allocation rules

Definition

Given a set of buyers B and a set of sellers S, an allocation rule φ maps each valuation profile $a \in \mathcal{A}^{B \times S}$ to a feasible outcome $\varphi(a) = (u(a), v(a); \mu(a))$ for (B, S, a).

Definition

Given a set of buyers B and a set of sellers S, an allocation rule φ is a stable rule if for each valuation profile $a \in \mathcal{A}^{B \times S}$, $\varphi(a) = (u(a), v(a); \mu(a))$ is a stable outcome for (B, S, a).

Definition

Given a set of buyers B and a set of sellers S, a rule $\varphi(a) = (u(a), v(a); \mu(a))$ such that $(u(a), v(a)) = (\underline{u}(a), \overline{v}(a))$ is a sellers-optimal stable rule.

Motivation

In van den Brink, Núñez and Robles (2021), regarding allocation rules for buyer-seller markets, were interested in:

- An axiomatic characterization of the buyers-optimal (and sellers-optimal) stable rules by means of some montonicity property.
    - In the ordinal setting (matching problems), Kojima and Manea (2010) prove that among the stable allocation rules, the deferred acceptance rule is the only one that satisfies weak Maskin monotonicity.
- The compatibility between stability and some sort of monotonicity.
- The compatibility between stability and some sort of fairness property.
- An axiomatic characterization of the fair-division rules.
- Until which extent these results can be extended to assignment markets with multiple partnership.

Axioms for the buyers-optimal stable rules

Definition

Given a set of buyers B and a set of sellers S, an allocation rule $\varphi \equiv (u, v; \mu)$ satisfies **buyer valuation monotonicity (BVM)** if for all $a, a' \in \mathcal{A}^{B\times S}$ and $t \in B$ such that $a'_{tj} \leq a_{tj}$ for all $j \in S$ and $a'_{ij} = a_{ij}$ for all $(i, j) \in (B \setminus \{t\}) \times S$,

$$(t, k) \in \mu(a) \cap \mu(a') \Rightarrow u_t(a') \leq u_t(a).$$

Theorem

On the domain of assignment markets with set of buyers B and set of sellers S, the buyers-optimal stable rules are the only stable rules that satisfy BVM.

Another axiomatization for the buyers-optimal stable rules

Definition

On the domain of assignment markets with set of buyers B and set of sellers S, an allocation rule $\varphi \equiv (u, v; \mu)$ is **buyers strategy proof (BSP)** if it is no manipulable by any group of buyers $B' \subseteq B$.

Theorem (Perez-Castrillo and Sotomayor, 2017)

On the domain of assignment markets with set of buyers B and set of sellers S, the buyers-optimal stable rules are the only stable rules that are BSP.

Pairwise monotonicity and the fair division rule

Definition

Given a set of buyers B and a set of sellers S, an allocation rule $\varphi \equiv (u, v; \mu)$ satisfies **pairwise monotonicity (PM)** if for all $a, a' \in \mathcal{A}^{B\times S}$ such that $a'_{ij} = a_{ij}$ for all $(i, j) \in B \times S \setminus \{(t, k)\}$ and $a'_{tk} \leq a_{tk}$,

$$u_t(a') \leq u_t(a) \text{ and } v_k(a') \leq v_k(a).$$

- The buyers-optimal rules (Núñez and Rafels, 2002),
- the sellers-optimal rules,
- the fair-division rules,
- the Shapley value ...

are pairwise monotonic.

Valuation fairness

Definition

Given a set of buyers B and a set of sellers S, an allocation rule $\varphi \equiv (u, v; \mu)$ satisfies **valuation fairness (VF)** if for all $a, a' \in \mathcal{A}^{B \times S}$ and $(t, k) \in B \times S$ such that $a'_{tk} \leq a_{tk}$ and $a'_{ij} = a_{ij}$ for all $B \times S \setminus \{(t, k)\}$, then

$$u_t(a') - u_t(a) = v_k(a') - v_k(a).$$

- van den Brink and Pintér (2015) characterize the Shapley value on the class of assignment games by means of submarket efficiency and valuation fairness.
- But all stable rules are submarket efficient...
- Hence, no stable rule satisfies VF.

The multiple-partners job market (Sotomayor, 1992)

- $F = \{f_1, f_2, \ldots, f_m\}$ a set of firms and $W = \{w_1, w_2, \ldots, w_n\}$ a set of workers.
- Each firm f_i values in $h_{ij} \geq 0$ being matched to worker w_j, who has a reservation value $t_j \geq 0$.
- If firm f_i hires worker w_j, there is a net value $a_{ij} = \max\{h_{ij} - t_j, 0\}$ to be shared.
- Each firm f_i may hire up to r_i workers and each worker w_j may work for up to s_j firms (capacities).
- We add a dummy agent on each side of the market: f_0 and w_0, with $a_{i0} = a_{0j} = a_{00} = 0$: $F_0 = F \cup \{0\}$ and $W_0 = W \cup \{0\}$.
- The multiple-partners job market is defined by (F, W, a, r, s).
- When all agents in $F \cup W$ have capacity one, we have the Shapley and Shubik assignment game (*simple assignment game*).

The multiple-partners assignment game

- A matching is a subset of $F_0 \times W_0$ such that each $f_i \in F$ is in exactly r_i pairs and each $w_j \in W$ is in exactly s_j pairs.
- A matching $\mu \in \mathcal{M}(F, W, r, s)$ is optimal if

$$\sum_{(f_i, w_j) \in \mu} a_{ij} \geq \sum_{(f_i, w_j) \in \mu'} a_{ij}, \text{ for all } \mu' \in \mathcal{M}(F, W, r, s)$$

- A coalitional game $(F \cup W, w_a)$ is defined, where, for all $T \subseteq F \cup W$,

$$w_a(T) = \max_{\mu \in \mathcal{M}(F \cap T, W \cap T, r, s)} \sum_{(f_i, w_j) \in \mu} a_{ij}.$$

- An outcome is $(u = (u_{ij})_{(f_i, w_j) \in \mu}, v = (v_{ij})_{(f_i, w_j) \in \mu}; \mu)$
- An outcome $(u, v; \mu)$ is feasible if for all $(f_i, w_j) \in \mu$,
  - $u_{ij} + v_{ij} = a_{ij}$, $u_{ij} \geq a_{i0}$, $v_{ij} \geq a_{0j}$,
  - if $f_i = f_0$, then $v_{0j} = a_{0j}$; if $w_j = w_0$, then $u_{i0} = a_{i0}$.
- A feasible outcome $(u, v; \mu)$ is stable if for al $(f_i, w_j) \notin \mu$, then

$$u_{ik} + v_{lj} \geq a_{ij} \text{ for all } (f_i, w_k) \in \mu \text{ and } (f_l, w_j) \in \mu.$$

Results in Sotomayor (1992, 1999, 2007)

- The set of stable outcomes is non-empty.
- If $(u, v; \mu)$ is a stable outcome and we define

$$U_i = \sum_{(f_i, w_j) \in \mu} u_{ij} \text{ and } V_j = \sum_{(f_i, w_j) \in \mu} v_{ij},$$

  then (U, V) is in the core of the coalitional game $(F \cup W, w_a)$.
- The set of stable outcomes is a lattice with a F-optimal stable outcome $(\overline{u}, \underline{v}; \mu)$ and a W-optimal stable outcome $(\underline{u}, \overline{v}; \mu)$.

Definition

A stable allocation rule is φ such that for all (F, W, a, r, s), $\varphi(a) = (u(a), v(a); \mu(a))$ is a stable outcome.

An example

$F = \{f_1, f_2, f_3\}$, $W = \{w_1, w_2, w_3\}$, $r_i = s_j = 2$ and

$$a = \begin{pmatrix} 4.5 & 20 & 4 \\ 5 & 3 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$

- Only one optimal matching that gives $w_a(F \cup W) = 36$.
- We associate this with a simple assignment game (Sotomayor, 1992): $\tilde{F}_1 = \{f_{11}, f_{12}, f_{21}, f_{22}, f_{31}, f_{32}\}$, $\tilde{W} = \{w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}\}$

$$\tilde{a} = \left(\begin{array}{cc|cc|cc} 4.5 & 4.5 & 20 & 0 & 0 & 0 \\ 4.5 & 4.5 & 0 & 0 & 4 & 0 \\ \hline 5 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 & 1 & 1 \\ \hline 0 & 2 & 3 & 3 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 & 2 \end{array}\right)$$

$\overline{u}_{12}(a) = \overline{u}_{11}(\tilde{a}) = 36 - 17.5 = 18.5$, $\overline{u}_{13}(a) = \overline{u}_{12}(\tilde{a}) = 36 - 32 = 4$.

An example (continuation)

We increase a_{11} and the optimal matching is the same:

$$a' = \begin{pmatrix} 4.6 & 20 & 4 \\ 5 & 3 & 1 \\ 2 & 3 & 2 \end{pmatrix} \quad \tilde{a} = \left(\begin{array}{cc|cc|cc} 4.6 & 4.6 & 20 & 0 & 0 & 0 \\ 4.6 & 4.6 & 0 & 0 & 4 & 0 \\ \hline 5 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 & 1 & 1 \\ \hline 0 & 2 & 3 & 3 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 & 2 \end{array}\right)$$

- Now, $\overline{u}_{12}(a') = \overline{u}_{11}(\tilde{a'}) = 36 - 17.6 = 18.4 < \overline{u}_{12}(a)$
- Since, $\overline{u}_{13}(a') = \overline{u}_{12}(\tilde{a'}) = 36 - 32 = 4 = \overline{u}_{13}(a)$, when we consider the total payoff of player f_1 we also get $\overline{U}_1(a') = 22.4 < 22.5 = \overline{U}_1(a)$.

Fact

1. *The firm-optimal stable rules are not firm-valuation monotonic.*
2. *The firm-optimal stable rules are not pairwise-monotonic.*

Firm covariance

Definition

A rule $\varphi \equiv (u, v; \mu)$ is firm-covariant (FC) if for all (F, W, a, r, s), all $f_{i_0} \in F$ and all $c \geq 0$ such that

1. $a^c_{i_0 j} = \max\{0, a_{i_0 j} - c\}\ \forall w_j \in W$ and $a^c_{ij} = a_{ij}\ \forall f_i \in F \setminus \{f_{i_0}\}$,
2. $c \leq a_{i_0 j}$ for all $(f_{i_0}, w_j) \in \mu$ and $\mu \in \mathcal{M}_a(F, W, r, s)$ and
3. $\mathcal{M}_a(F, W, r, s) \subseteq \mathcal{M}_{a^c}(F, W, r, s)$, then

$u_{i_0 j}(a^c) = u_{i_0 j}(a) - c$ for all $(f_{i_0}, w_j) \in \mu$ and $u_{ij}(a^c) = u_{ij}(a)$ otherwise.

$c \leq c^* = \min\{c \geq 0 \mid \exists \mu \in \mathcal{M}_{a^c}(F, W, r, s)\, a^c_{ij} = 0$ for some $(f_i, w_j) \in \mu\}$

Theorem

1. The firm-optimal stable rule is the only stable rule that is firm-covariant.

2. The worker-optimal stable rule is the only stable rule that is worker-covariant.

Corollary The firm-optimal stable rule is weak firm-valuation monotonic.

Manipulability of the optimal stable rules

The firm-optimal stable rule is manipulable:

Example (Pérez-Castrillo and Sotomayor, 2017):

$F = \{f_1, f_2\}$, $r_1 = 2$, $r_2 = 1$, $W = \{w_1, w_2, w_3\}$, $s_1 = s_2 = s_3 = 1$.
$h_1 = (7, 6, 4)$, $h_2 = (8, 6, 3)$ and $t_1 = t_2 = t_3 = 0$

$$a = \begin{pmatrix} 7 & \mathbf{6} & \mathbf{4} \\ \mathbf{8} & 6 & 3 \end{pmatrix} \quad \tilde{a} = \left(\begin{array}{c|c|c} 7 & 6 & 0 \\ 7 & 0 & 4 \\ \hline 8 & 6 & 3 \end{array}\right)$$

$$\overline{U}_1(a) = (a_{12} - \underline{v}_{12}(a)) + (a_{13} - \underline{v}_{13}(a)) = (6-1) + (4-0) = 9.$$

If f_1 reports $h'_1 = (8, 7, 7)$, the optimal matching does not change and

$$\overline{U}_1(a') = (a_{12} - \underline{v}_{12}(a')) + (a_{13} - \underline{v}_{13}(a')) = (6-0) + (4-0) = 10.$$

A weaker non-manipulability property

Definition

Let (F, W, r, s), a firm $f_{i_0} \in F$ manipulates a rule $\varphi \equiv (v; \mu)$ by constantly over-reporting its valuations if there exist valuations (h, t) and $c > 0$ such that f_{i_0} gets a higher payoff at $(v(h', t); \mu(h', t))$ than at $(v(h, t); \mu(h, t))$, where $h'_{i_0 j} = h_{i_0 j} + c$ for all $w_j \in W$ and $h'_{ij} = h_{ij}$ otherwise.

Fact

On the domain of multiple-partners job markets where all firm-worker pairs are mutually acceptable ($h_{ij} \geq t_j$ for all (f_i, w_j)),

1. *No firm can manipulate the firm-optimal stable rule by constantly over-reporting its valuations.*
2. *No worker can manipulate the worker-optimal stable rule by under-reporting his/her reservation value.*

The fair division rules

$\varphi^\tau \equiv (u^\tau, v^\tau; \mu)$ si a fair division rule if for all $(f_i, w_j) \in \mu$,

$$u_{ij}^\tau(a) = \frac{1}{2}\overline{u}_{ij}(a) + \frac{1}{2}\underline{u}_{ij}(a) \text{ and } v_{ij}^\tau(a) = \frac{1}{2}\overline{v}_{ij}(a) + \frac{1}{2}\underline{v}_{ij}(a).$$

Definition

A rule $\varphi \equiv (u, v; \mu)$ satisfies great valuation fairness (GVF) if for all (F, W, a, r, s) and all $c \geq 0$ such that

1. $a_{ij}^c = \max\{0, a_{ij} - c\}$ for all $f_i \in F$ and $w_j \in W$,
2. $c \leq a_{ij}$ for all $(f_i, w_j) \in \mu$ and $\mu \in \mathcal{M}_a(F, W, r, s)$ and
3. $\mathcal{M}_a(F, W, r, s) \subseteq \mathcal{M}_{a^c}(F, W, r, s)$, then

$$u_{ij}(a^c) - u_{ij}(a) = v_{ij}(a^c) - v_{ij}(a) \text{ for all } (f_i, w_j) \in \mu.$$

The derived assignment game with multiple partnership

Definition

Let (F, W, a, r, s), μ an optimal matching, $T = \{f_{i_0}, w_{j_0}\}$ with $(f_{i_0}, w_{j_0}) \in \mu$ such that $a_{i_0 j_0} = a_{i_0 0} + a_{0 j_0}$ and $z = (u, v; \mu)$ stable. The derived assignment market at T and z is $(F^T, W^T, a^{T,z}, r^T, s^T)$:

$$F^T = \begin{cases} F \setminus \{f_{i_0}\} & \text{if } r_{i_0} = 1, \\ F & \text{otherwise} \end{cases} \quad W^T = \begin{cases} W \setminus \{w_{j_0}\} & \text{if } s_{j_0} = 1, \\ W & \text{otherwise} \end{cases},$$

$a^{T,z}_{ij} = a_{ij}$ for all $f_i \in F^T,\ w_j \in W^T$,

$$\begin{array}{rcl} (i)\ a^{T,z}_{k0} & = & \max\{a_{k0}, a_{kj_0} - v_{ij_0}\}, \text{ for all } f_k \in F^T, \\ (ii)\ a^{T,z}_{0k} & = & \max\{a_{0k}, a_{i_0 k} - u_{i_0 j}\}, \text{ for all } w_k \in W^T, \end{array}$$

and $r^T_{i_0} = r_{i_0} - 1$ if $f_{i_0} \in F^T$, $r^T_k = r_k$ otherwise,
$s^T_{j_0} = s_{j_0} - 1$ if $w_{j_0} \in W^T$, $s^T_k = s_k$ otherwise.

Axiomatization of the fair division rules

Definition

On the domain of multiple-partners job markets, a stable allocation rule φ is weak derived consistent (WDC) if for all (F, W, a, r, s) and all $T = \{f_i, w_j\}$ with $(f_i, w_j) \in \mu$ and $a_{ij} = a_{i0} + a_{0j}$, it holds

$$
\begin{array}{ll}
(i) & \mu' = \mu \setminus \{(f_i, w_j)\} \text{ is optimal for } (F^T, W^T, a^{T,(u,v)}, r^T, s^T), \\
(ii) & u_{kl}(F^T, W^T, a^{T,(u,v)}, r^T, s^T) = u_{kl}(F, W, a, r, s) \text{ for all } (f_k, w_l) \in \mu' \\
(iii) & v_{kl}(F^T, W^T, a^{T,(u,v)}, r^T, s^T) = v_{kl}(F, W, a, r, s) \text{ for all } (f_k, w_l) \in \mu'
\end{array}
$$

where $\varphi(F, W, a, r, s) = (u, v; \mu)$.

Theorem

On the domain of multiple-partners job markets, the fair division rules are the only stable rules that satisfy GVF and WDC.

Further research

- Not much is known about the core of the multiple-partners game:
    - It is an open question the existence of an optimal core allocation for each side of the market.
    - It is known (Sotomayor, 2002, 2007) that a worst core allocation for any side of the market may not exist.
- This is why we focus on the set of (pairwise)-stable payoff vectors.
    - Extreme stable payoff vectors could be analyzed.

Thank you