Sharing the Cost of a Gas Distribution Network.

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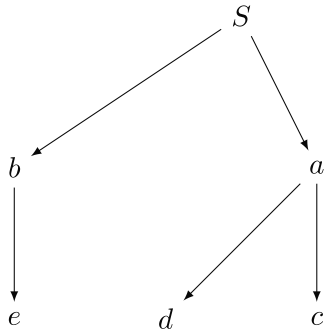
Overview

1. The model;
   - Consumers;
   - Demands;
   - Network;
   - Cost function.
2. Cost sharing rules;
   - 3 rules;
   - Axiomatic characterization.
3. Multi-choice cooperative games.
   - Values and Rules;
   - Core and Rules.

The Model

The model: (N, P)



Gas distribution network P

-$N = \{a, b, \ldots, n\}$ finite set of **consumers**.

-Consumers are linked to a (unique) source S via **pipelines**.

-Consumers and pipelines form a **gas distribution network**, represented by a directed tree P.

The model: (N, P, q)

-Each consumer $i \in N$ has an **maximal demand** $q_i \in \mathbb{N}$ and is endowed with the discrete set of **available demands**

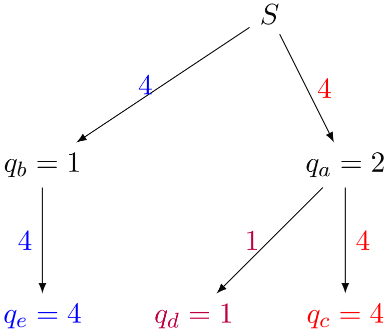
$$M_i = \{0, 1, 2, \ldots, q_i\}.$$

-The profile of maximal demands is denoted by $q = (q_a, \ldots, q_n)$.

<u>NB:</u> $1 \leq q_i \leq q_n$, for all $i \in N$.

The model: (N, P, q)

Since the network operator must be able to meet any maximal demand, each pipeline must be large enough to meet any maximal downstream demand.



-The **Cost function** can evaluate the cost of any pipeline of any size

$$C : N \times \{1, \dots, q_n\} \to \mathbb{R}_+,$$

-The cost of the i-th pipeline when sized to meet a demand of j is given by

$$C(i, j) \in \mathbb{R}_+.$$

<u>NB:</u> $C(i, 0) = 0$ and $C(i, j) \leq C(i, j + 1)$, for all $j < q_n$.

The model: (N, P, q, C)

An incremental cost A_{ij}^C, $i \in N$, $j \leq \overline{q}_i$, is defined as

$$A_{ij}^C = C(i, j) - C(i, j-1).$$

C	a	b	c	d	e
1	5	2	7	4	5
2	10	8	13	9	11
3	15	12	16	13	15
4	20	15	22	17	20

$$\begin{aligned} A_{a3}^C &= C(a,3) - C(a,2) \\ &= 15 - 10 \\ &= 5. \end{aligned}$$

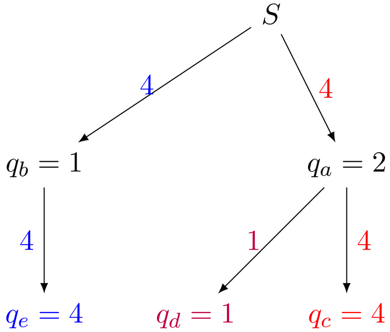
NB: An incremental cost A_{ij}^C can be interpreted as the cost of upgrading the pipeline i from a size $j-1$ to a size j.

The model: (N, P, q, C)

The **total cost** is computed as the sum of the cost of each pipeline. Each pipeline is large enough to meet any maximum downstream demand.

$$\sum_{i \in N} C(i, \overline{q}_i), \quad \text{t.q.} \quad \overline{q}_i = \max_{k \in \hat{P}(i) \cup i} q_k.$$

The model



C	a	b	c	d	e
1	5	2	7	4	5
2	10	8	13	9	10
3	15	12	16	13	15
4	20	15	22	17	20

$$\text{Total} = 81.$$

Cost Sharing Rules

Rules

-A gas distribution problem is denoted by (N, P, q, C) or (N, P, q, A^C). The class of all problems is denoted by G.
-A (cost sharing) rule is a map

$$f : \mathrm{G} \to \mathbb{R}_+^{\sum_{i \in N} q_i}.$$

-It describes how much each consumer has to pay for each of their available demands.
-It recovers the total cost of operating the network.

Rules

Recall: $q_a = 2$, $q_b = 1$, $q_c = 4$, $q_d = 1$ et $q_e = 4$.

f	a	b	c	d	e
1	f_{a1}	f_{b1}	f_{c1}	f_{d1}	f_{e1}
2	f_{a2}	x	f_{c2}	x	f_{e2}
3	x	x	f_{c3}	x	f_{e3}
4	x	x	f_{c4}	x	f_{e4}

Connection principle:
" Consumers should only pay for the portion of the network they use. "

Uniformity principle:
" Two consumers with the same demands should be charged the same amount. "

Independence principle:
"A consumer should not be charged for costs generated by demands higher than its own. "

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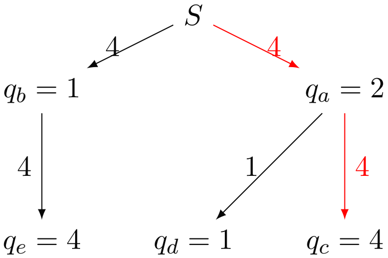
The Connection rule Ψ is defined, for each $(N, P, q, A^C) \in \mathrm{G}$, by

$$\forall i \in N, j \leq q_i, \quad \Psi_{ij}(N, P, q, A^C) = \sum_{k \in \hat{P}^{-1}(i) \cup i} \frac{A^C_{kj}}{(\hat{P}(k) \cup k) \cap Q(j)}.$$

-$\hat{P}^{-1}(i)$ is the set of pipelines located upstream of i.
-$Q(j) = \{k \in N : q_k \geq j\}$.
-$\hat{P}(k)$ is the set of pipelines located downstream of k.

Connection rule

A^C	a	b	c	d	e
1	5	2	7	4	5
2	5	6	6	5	5
3	5	4	3	4	5
4	5	3	6	4	5



$$\text{c's share} = \Psi_{c1} + \Psi_{c2} + \Psi_{c3} + \Psi_{c4}$$

$$\Psi_{c1} = A^C_{c1} + \frac{A^C_{a1}}{3} = 5.7$$

$$\Psi_{c2} = A^C_{c2} + \frac{A^C_{a2}}{2} = 8.5$$

$$\Psi_{c3} = A^C_{c3} + A^C_{a3} = 8$$

$$\Psi_{c4} = A^C_{c4} + A^C_{a4} = 11$$

Axiom (Weak linearity)

For each $(N, P, q, A^C), (q, A^{C'}) \in \mathrm{G}$ *and* $\beta \in \mathbb{R}_+$,

$$f(q, A^C + \beta A^{C'}) = f(q, A^C) + \beta f(q, A^{C'}).$$

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Axiom (Independence to higher demands)

For each $(N, P, q, A^C) \in \mathrm{G}$ *and each* $l \leq q_n$,

$$\forall (i,j) \in M^+ : j \leq l, \quad f_{ij}(q, A^C) = f_{ij}((l \wedge q_k)_{k \in N}, A^C).$$

NB: $l \wedge q_k = \max\{l; q_k\}$

Pick any $i \in N$, $j \leq q_n$, the **unit cost matrix** I^{ij} is defined as

$$\forall k \in N, l \leq q_n, \quad I^{ij}_{kl} = \begin{cases} 1 & \text{if } k = i, l = j, \\ 0 & \text{otherwise.} \end{cases}$$

This matrix isolates the incremental cost generated by the j-th upgrade of pipeline i.
We will use it to analyze how the cost shares of the consumers behave regarding a given incremental cost.

Axiom (Independence to irrelevant costs)

For each $(N, P, q, I^{ij}) \in \mathrm{G}$,

$$\forall h \in Q(j), h \notin (\hat{P}(i) \cup \{i\}), \quad f_{hj}(q, I^{ij}) = 0.$$

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Axiom (Downstream symmetry)

For each $(N, P, q, I^{ij}) \in \mathrm{G}$,

$$\forall h, h' \in [\hat{P}(i) \cup \{i\}] \cap Q(j), \quad f_{hj}(q, I^{ij}) = f_{h'j}(q, I^{ij}).$$

A rule f on G satisfies Weak linearity, Independence to higher demands, Independence to irrelevant costs and Downstream symmetry if and only if $f = \Psi$.

Uniformity principle:
" Two consumers with the same demands should be charged the same amount. "

Independence principle:
"A consumer should not be charged for costs generated by demands higher than its own. "

Uniform rule

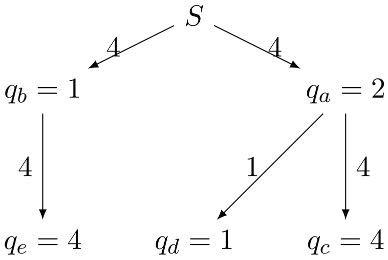
The Uniform rule Υ is defined, for each $(N, P, q, A^C) \in \mathrm{G}$, by

$$\forall i \in N, \forall j \leq q_i, \quad \Upsilon_{ij}(q, A^C) = \frac{1}{|Q(j)|} \sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A^C_{kj}.$$

-$Q(j) = \{k \in N : q_k \geq j\}$.
-$\hat{P}^{-1}(Q(j))$ is the set of pipelines located upstream of the consumers in $Q(j)$.

Uniform rule

A^C	a	b	c	d	e
1	5	2	7	4	5
2	5	6	6	5	5
3	5	4	3	4	5
4	5	3	6	4	5



$$\text{c's share} = \Upsilon_{c1} + \Upsilon_{c2} + \Upsilon_{c3} + \Upsilon_{c4}$$

$$\Upsilon_{c1} = \frac{1}{5}(A^C_{a1} + A^C_{b1} + A^C_{c1} + A^C_{d1} + A^C_{e1}) = 4.6$$

$$\Upsilon_{c2} = \frac{1}{3}(A^C_{a2} + A^C_{b2} + A^C_{c2} + A^C_{e2}) = 7.3$$

$$\Upsilon_{c3} = \frac{1}{2}(A^C_{a3} + A^C_{b3} + A^C_{c3} + A^C_{e3}) = 8.5$$

$$\Upsilon_{c4} = \frac{1}{2}(A^C_{a4} + A^C_{b4} + A^C_{c4} + A^C_{e4}) = 9.5$$

Axiom (Non-increasing inequalities)

For each $(N, P, q, A^C), (N, P, q, A^{C'}) \in \mathrm{G}$ *such that* $A^{C'}_{ij} \geq A^C_{ij}$, *for each* $i \in N$ *and* $j \leq q_n$,

$$\forall j \in \{1, \ldots, q_n\},$$
$$\max_{i \in Q(j)} f_{ij}(q, A^{C'}) - \min_{i \in Q(j)} f_{ij}(q, A^{C'})$$
$$\leq \max_{i \in Q(j)} f_{ij}(q, A^C) - \min_{i \in Q(j)} f_{ij}(q, A^C).$$

Theorem 2

A rule f on G satisfies Independence to higher demands and Non-increasing inequalities if and only if $f = \Upsilon$.

Connection principle VS **Uniformity Principle**

Mixed rules

Ψ	a	b	c	d	e
1	1.7	1	8.7	5.7	6
2	2.5	x	8.5	x	11
3	x	x	8	x	9
4	x	x	11	x	8

Υ	a	b	c	d	e
1	4.6	4.6	4.6	4.6	4.6
2	7.3	x	7.3	x	7.3
3	x	x	8.5	x	8.5
4	x	x	9.5	x	9.5

Pick any $\alpha \in [0,1]^{q_n}$. The Mixed rule μ^α is defined, for each $(N, P, q, A^C) \in \mathrm{G}$, by

$$\forall i \in N, \forall j \leq q_i, \quad \mu^\alpha_{ij}(q, A^C) = \alpha_j \Psi_{ij}(q, A^C) + (1-\alpha_j)\Upsilon_{ij}(q, A^C).$$

Mixed Rules

Pick $\alpha = (1, 0.8, 0.5, 0)$.

Ψ	a	b	c	d	e
1	1.7	1	8.7	5.7	6
2	2.5	x	8.5	x	11
3	x	x	8	x	9
4	x	x	11	x	8

Υ	a	b	c	d	e
1	4.6	4.6	4.6	4.6	4.6
2	7.3	x	7.3	x	7.3
3	x	x	8.5	x	8.5
4	x	x	9.5	x	9.5

μ^α	a	b	c	d	e
1	1.7	1	8.7	5.7	6
2	3, 46	x	8, 26	x	10, 26
3	x	x	8.25	x	8.75
4	x	x	9.5	x	9.5

Axiom (Equal impact of irrelevant costs)

For each $(q, I^{ij}) \in \mathrm{G}$,

$$\forall h, h' \in Q(j), h, h' \notin \hat{P}(i) \cup \{i\}, \quad f_{hj}(q, I^{ij}) = f_{h'j}(q, I^{ij}).$$

Axiom (Equal impact of irrelevant costs)

For each $(q, I^{ij}) \in \mathrm{G}$,

$$\forall h, h' \in Q(j), h, h' \notin \hat{P}(i) \cup \{i\}, \quad f_{hj}(q, I^{ij}) = f_{h'j}(q, I^{ij}).$$

Axiom (Location independence of irrelevant costs)

For each $(N, P, q, I^{ij}), (N, P, q, I^{i'j}) \in \mathrm{G}$,

$$\forall h \in Q(j), h \notin \left[\hat{P}(i) \cup \{i\}\right] \cup \left[\hat{P}(i') \cup \{i'\}\right], \quad f_{hj}(q, I^{ij}) = f_{hj}(q, I^{i'j}).$$

Axiomatic characterization

Axiom (Fairness)

For each $(N, P, q, I^{ij}) \in \mathrm{G}$,

$$\forall k \in [\hat{P}(i) \cup \{i\}] \cap Q(j), \forall h \in Q(j), \quad f_{kj}(q, I^{ij}) \geq f_{hj}(q, I^{ij}).$$

A rule f on G satisfies Weak linearity, Independence to higher demands, Equal impact of irrelevant costs, Location independence of irrelevant costs and Fairness if and only if $f = \mu^{\alpha}$, for some $\alpha \in [0,1]^{q_n}$.

Multi-Choice Games

A multi-choice game is given by:

- A finite player set $N = \{a, \ldots, n\}$;
- For each $i \in N$, a finite set $M_i = \{0, \ldots, q_i\}$;
- A coalition is a profile $s = (s_a, \ldots, s_n) \in \prod_{i \in N} M_i$, $q = (q_1, \ldots, q_n)$ is the grand coalition;
- A characteristic function

$$v : \prod_{i \in N} M_i \to \mathbb{R};$$

- A (multi-choice) game is denoted (q, v), the full class of multi-choice games is denoted $\mathcal{G}$;

- Denote by M^+ the set of all (i, j) where $i \in N$ and $j \in M_i \setminus \{0\}$.
- A payoff vector x is an element of $\mathbb{R}^{|M^+|}$. For each $(i, j) \in M^+$, $x_{ij} \in \mathbb{R}$ specifies a payoff for the activity level j of player i.
- A value is a map
$$f : \mathcal{G} \to \mathbb{R}^{|M^+|}.$$

For each game $(q, v) \in \mathcal{G}$, the multi-choice Shapley value is defined as

$$\forall (i,j) \in M^+, \quad \varphi_{ij}(q,v) = \sum_{\substack{s \in \prod_{i \in N} M_i \\ (i,j) \in T(s)}} \frac{\Delta_v(s)}{|T(s)|}.$$

where

$$\Delta_v(s) = v(t) - \sum_{t \leq s, t \neq s} \Delta_v(t)$$

$$T(s) = \Big\{ (i, s_i) \in M^+ : s_i \geq s_k, \; \forall k \in N \Big\}.$$

For each game $(q, v) \in \mathcal{G}$, the multi-choice Equal division value is defined as

$$\forall (i,j) \in M^+,$$
$$\xi_{ij}(q,v) = \frac{1}{|Q(j)|}\Big[v((j \wedge q_k)_{k\in N}) - v((((j-1) \wedge q_k)_{k\in N}))\Big].$$

$Q(j) = \{i \in N : q_i \geq j\}$.

For each game $(q, v) \in \mathcal{G}$, the multi-choice Equal division value is defined as

$$\forall (i, j) \in M^+,$$
$$\xi_{ij}(q, v) = \frac{1}{|Q(j)|}\Big[v((j \wedge q_k)_{k\in N}) - v((((j-1) \wedge q_k)_{k\in N}))\Big].$$

Pick any $\alpha \in [0, 1]^{q_n}$. For each $(q, v) \in \mathcal{G}$, the multi-choice Egalitarian Shapley value χ^α is defined as

$$\forall (i, j) \in M^+, \quad \chi^\alpha_{ij}(q, v) = \alpha_j \varphi_{ij}(q, v) + (1 - \alpha_j)\xi_{ij}(q, v).$$

For each $(N, P, q, A^C) \in \mathrm{G}$, the associated gas distribution (multi-choice) game $(q, v^{C,P})$ is defined as

$$\forall s \leq q, \quad v^{C,P}(s) = \sum_{i \in N} C(i, \overline{s}_i),$$

where $\forall i \in N, \quad \overline{s}_i = \max_{k \in \hat{P}(i) \cup i} s_k$.

$v^{C,P}(s)$ is the total cost of a hypothetical gas distribution problem in which s is the profile of effective demands.

For each $(N, P, q, A^C) \in \mathrm{G}$,

$$\varphi(q, v^{C,P}) = \Psi(q, A^C)$$
$$\xi(q, v^{C,P}) = \Upsilon(q, A^C)$$
$$\chi^\alpha(q, v^{C,P}) = \mu^\alpha(q, A^C)$$

The Core of a multi-choice game $(q, v) \in \mathcal{G}$ ([Grabisch and Xie, 2007]) is denoted by $Co(q, v)$ and is defined as

$$x \in Co(q,v) \iff \begin{cases} \forall s \leq q, \quad \sum_{i \in N} \sum_{j=1}^{s_i} x_{ij} \leq v(s) \\ \forall h \leq q_n, \quad \sum_{i \in N} \sum_{j=1}^{h \wedge q_i} x_{ij} = v((h \wedge q_i)_{i \in N}). \end{cases}$$

[Lowing and Techer, 2021] show that for each super-modular game $(q, v) \in \mathcal{G}$,

$$\varphi(q, v) \in Co(q, v).$$

NB: A game $(q, v) \in \mathcal{G}$ is super-modular if $v(s \vee t) + v(s \wedge t) \geq v(s) + v(t)$ for each $s, t \leq q$.

We show that $(q, v^{C,P})$ is super-modular, therefore

$$\varphi(q, v^{C,P}) \in Co(q, v^{C,P})$$

Thank You !

References

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