Core stability and other applications of minimal balanced collections

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Notation

- Let $N = \{1, \dots, n\}$ be a set of *players*,
- Denote by 2^N the power set of N,
- A (TU) game (N, v) is a pair consisting of the set N and a mapping $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$, called the *characteristic function*,
- We call the nonempty subsets of N *coalitions*.

Preimputations

- Denote by $x(S)$ the sum $\sum_{i \in S} x_i$.
- Denote by $\mathbb{R}^N$ the set of n-dimensional vectors, called *payoff vectors*.

Definition

$X(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$ is called the set of *preimputations*.

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- We also define the set of imputations as

$$I(N, v) = \{x \in X(N, v) \mid x_i \geq v(\{i\})\}.$$

Domination

Consider x, y two preimputations, and $S \subseteq N$ a coalition.

Definition

We say that x *dominates* y via S, denoted $x \operatorname{dom}_S y$, if

$$x(S) \leq v(S) \quad \text{and} \quad x_i > y_i, \text{ for all } i \in S.$$

We say that $x \operatorname{dom} y$ if there exists a coalition S such that $x \operatorname{dom}_S y$.

The stable sets and the core

- The stable sets (von Neumann & Morgenstern[1], 1944).

Definition

We say that a subset U of $I(N, v)$ is a *stable set* if

- *(external stability)* $\forall y \notin U, \exists x \in U$ such that $x \,\mathrm{dom}\, y$;
- *(internal stability)* $x \,\mathrm{dom}\, y$ & $y \in U \implies x \notin U$.

[1] Von Neumann, J., and Morgenstern, O., (1944). *Theory of Games and Economic Behavior*. Princeton university press.

[2] Gillies, D. B. (1959). 3. Solutions to general non-zero-sum games. In *Contributions to the Theory of Games (AM-40), Volume IV* (pp. 47-86). Princeton University Press.

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- *(internal stability)* $x \operatorname{dom} y \;\&\; y \in U \implies x \notin U$.

- The core (popularized by Gillies[2], 1959)

Definition

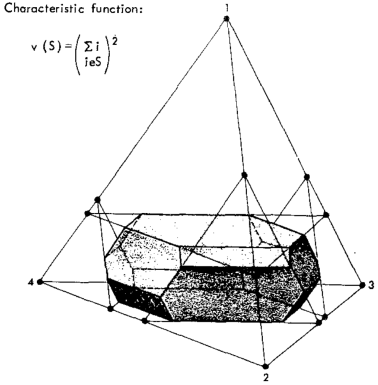
Let (N, v) be a game. The *core* of (N, v) is defined by

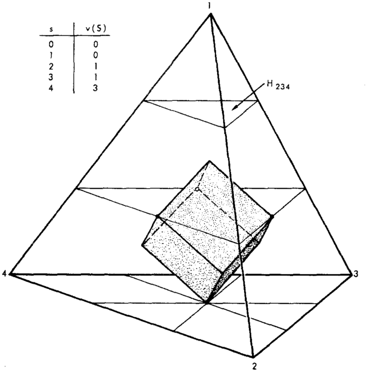
$$C(N, v) = \{x \in X(N, v) \mid x(S) \geq v(S),\ \forall S \subseteq N\}.$$

[1] Von Neumann, J., and Morgenstern, O., (1944). *Theory of Games and Economic Behavior*. Princeton university press.

[2] Gillies, D. B. (1959). 3. Solutions to general non-zero-sum games. In *Contributions to the Theory of Games (AM-40), Volume IV* (pp. 47-86). Princeton University Press.

Graphical representations of 4-player game's core





Relations between the core and stable sets

Theorem

The core is included in every stable set.

Proof.

The core contains only undominated imputations. Then, it must be included in every stable set.

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Under which conditions is the core stable?

Games with a stable core

- Convex games (Shapley[1], 1971)
- Games with a large core (Sharkey[2], 1982)
- Extendable balanced games (Kikuta and Shapley[3], 1986)
- Vital-exact extendable balanced games (Shellshear and Sudhölter[4], 2009).

[1] Shapley, L. S. (1971). Cores of convex games. *International Journal of Game Theory, 1*(1), 11-26

[2] Sharkey, W. W. (1982). Cooperative games with large cores. *International Journal of Game Theory, 11*(3-4), 175-182

[3] Kikuta, K., and Shapley, L. S. (1986). Unpublished manuscript.

[4] Shellshear, E., and Sudhölter, P. (2009). On core stability, vital coalitions, and extendability. *Games and Economic Behavior, 67*(2), 633-644

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Moreover, core stability and vital-exact extendability are equivalent for

- matching games,
- simple flow games,
- minimum coloring games.

[1] Shapley, L. S. (1971). Cores of convex games. *International Journal of Game Theory, 1*(1), 11-26

[2] Sharkey, W. W. (1982). Cooperative games with large cores. *International Journal of Game Theory, 11*(3-4), 175-182

[3] Kikuta, K., and Shapley, L. S. (1986). Unpublished manuscript.

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Balanced collections

Denote by $\mathbb{1}^T$ the n-dimensional $(0,1)$-vector such that $\mathbb{1}^T_i = 1$ iff $i \in T$.

Definition

Let $\mathcal{B} \subseteq 2^N$ be a collection of coalitions. We say that $\mathcal{B}$ is *balanced* if there exists a balancing vector $(\lambda^{\mathcal{B}}_S)_{S\in\mathcal{B}}$ such that

$$\sum_{S\in\mathcal{B}} \lambda^{\mathcal{B}}_S \mathbb{1}^S = \mathbb{1}^N.$$

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Examples with three players:

- $\mathcal{B}^1 = \{\overline{1}, \overline{2}, \overline{3}\}$ with $\lambda^{\mathcal{B}^1} = (1,1,1)$
- $\mathcal{B}^2 = \{\overline{12}, \overline{13}, \overline{23}\}$ with $\lambda^{\mathcal{B}^2} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

Example with four players:

- $\mathcal{B}^3 = \{\overline{12}, \overline{13}, \overline{14}, \overline{234}\}$ with $\lambda^{\mathcal{B}^3} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

Minimal balanced collections

Definition

A balanced collection is *minimal* if and only if it does not contain a proper subcollection that is balanced.

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Theorem (Bondareva-Shapley, sharp form)

A game (N, v) has a nonempty core if and only if for any minimal balanced collection $\mathcal{B}$ with balancing vector $(\lambda_S^{\mathcal{B}})_{S\in\mathcal{B}}$, we have

$$\sum_{S\in\mathcal{B}} \lambda_S^{\mathcal{B}} v(S) \leq v(N).$$

Moreover, none of the inequalities is redundant, except the one for $\mathcal{B} = \{N\}$.

Peleg's method[1]

Some notation:

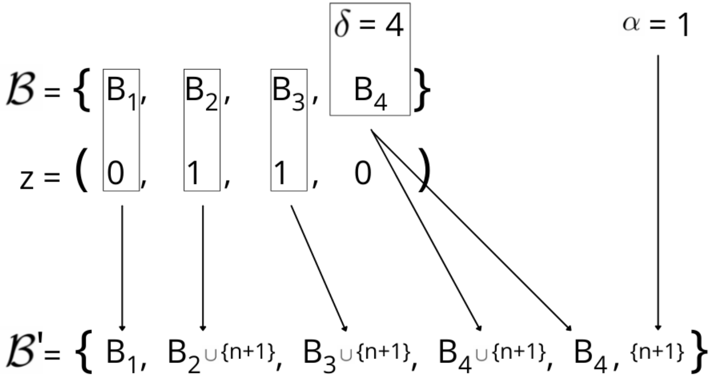
- Let $\mathcal{B} = \{B_1, \dots, B_k\}$ be a balanced collection over N with k coalitions. We denote its balancing vector by $\lambda^{\mathcal{B}}$.
- We call an element $z \in \{0,1\}^k$ an *extension vector* of $\mathcal{B}$.
- Let δ be an integer such that $0 \le \delta \le k$ and α an integer $\alpha \in \{0,1\}$ that we call *doubling index* and *adding index* respectively.

From this extension vector and these indices, we can construct an *extension* of $\mathcal{B}$, denoted by $\mathcal{B}'_{z,\delta,\alpha}$, that is a collection of coalitions on the ground set $N' = N \cup \{n+1\}$.

[1] Peleg, B. (1965). An inductive method for constructing mimmal balanced collections of finite sets. *Naval Research Logistics Quarterly*, 12(2).

Peleg's method

This extension is constructed as follows:



Peleg's theorem

Theorem (Peleg, 1965)

The extension $\mathcal{B}'_{z,\delta,\alpha}$ on $N' = N \cup \{n+1\}$ is a minimal balanced collection if and only if one of the following conditions is satisfied:

- $\mathcal{B}$ is a minimal balanced collection on N, $\alpha = 1$, $\delta = 0$ and $\langle \lambda^{\mathcal{B}}, z \rangle < 1$;
- $\mathcal{B}$ is a minimal balanced collection on N, $\alpha = 0$, $\delta \neq 0$ and

$$1 > \langle \lambda^{\mathcal{B}}, z \rangle > 1 - \lambda^{\mathcal{C}}_{\delta};$$

- $\mathcal{B}$ is a minimal balanced collection on N, $\alpha = 0$, $\delta = 0$ and $\langle \lambda^{\mathcal{B}}, z \rangle = 1$;
- $\mathcal{B}$ is the union of two minimal balanced collections on N, the rank of the adjacency matrix $A^{\mathcal{B}}$ is $k - 1$, and there exists a unique w such that $\langle \lambda^{\mathcal{B}}, z \rangle = 1$.

Computation of the minimal balanced collections

Players	Minimal balanced collections	CPU time (seconds)
3	6	0.0005
4	42	0.0057
5	1292	0.23
6	201 076	44
7	?	$>$ 38 hours (estimation)

Applications of balanced collections

Thanks to the balanced collections, we can compute/check:

- nonemptiness of the core (Bondareva-Shapley);
- the set of effective coalitions;
- the set of exact coalitions;
- the set of vital coalitions;
- the set of strictly vital-exact coalitions;
- the set of feasible collections;

$\Longrightarrow$ **and the stability of the core.**

Effective coalitions

Definition

We say that a coalition S is *effective* if $\forall x \in C(N, v)$, $x(S) = v(S)$.
We denote by $\mathcal{E}(N, v)$ the set of effective coalitions.

Proposition

$\mathcal{E}(N, v)$ is the union of all the minimal balanced collections $\mathcal{B}$ such that

$$\sum_{S \in \mathcal{B}} \lambda_S v(S) = v(N).$$

Strictly vital-exact coalitions

Definition

We say that a coalition S is *strictly vital-exact* if there exists $x \in C(N, v)$ such that $x(S) = v(S)$ and $x(T) > v(T)$, for all $T \in 2^S \setminus \{\emptyset, S\}$.
We denote by $\mathcal{VE}$ the set of strictly vital-exact coalitions.

Proposition

Let (N, v) be a balanced game. The core is stable only if

$$C(N, v) = \{x \in X(N, v) \mid x(S) \geq v(S), \forall S \in \mathcal{VE}\}.$$

Strictly vital-exact coalitions

Denote by v^S the game that only differs from v by

$$v^S(N \setminus S) = v(N) - v(S).$$

Proposition

A coalition $S \in 2^N \setminus \{\emptyset, N\}$ is strictly vital-exact if and only if there exists $x \in C(N, v)$ such that $x(S) = v(S)$ and

$$\mathcal{E}(N, v^S) \subseteq \{R \in 2^N \mid R \cap (N \setminus S) \neq \emptyset\}.$$

Feasible collections

Take $\mathcal{S} \subseteq 2^N$. We define the *region* $X_{\mathcal{S}}$ associated to $\mathcal{S}$ as

$$X_{\mathcal{S}} = \{x \in X(N, v) \mid x(S) < v(S) \iff S \in \mathcal{S}\}.$$

Definition

We say that $\mathcal{S}$ is *feasible* if the region $X_{\mathcal{S}}$ is nonempty.

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Definition

We say that $\mathcal{S}$ is *feasible* if the region $X_{\mathcal{S}}$ is nonempty.

Denote $\mathcal{S}^c = \{N \setminus S \mid S \in \mathcal{S}\}$.

Proposition

Let (N, v) be a balanced game and $\mathcal{S} \subseteq \mathcal{VE}$. $\mathcal{S}$ is feasible if and only if for every minimal balanced collections $\mathcal{B}$ of $(\mathcal{VE} \setminus \mathcal{S}) \cup \mathcal{S}^c$, we have

$$\sum_{T \in \mathcal{B}} \lambda_S^{\mathcal{B}} v^{\mathcal{S}}(T) \leq v(N)$$

with strict inequality if $\mathcal{B} \cap \mathcal{S}^c \neq \emptyset$.

Nested balancedness[1], 2021

Theorem

Let (N, v) be a balanced game. Then (N, v) has a stable core if and only if for every feasible collection $\mathcal{S}$ and every $(\mathcal{B}_S)_{S\in\mathcal{S}} \in \mathbb{C}(\mathcal{S})$, either

$$\exists Z' \in \mathbb{B}(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}) \setminus \mathbb{B}_0(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}) : \sum_{z\in Z'} \delta_z^{Z'} a_z > v(N) \text{ holds or}$$

$$\exists Z' \in \mathbb{B}_0(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}) : \sum_{z\in Z'} \delta_z^{Z'} a_z \geq v(N) \text{ holds.}$$

[1] Grabisch, M., & Sudhölter, P. (2021). Characterization of TU games with stable core by nested balancedness. *Mathematical Programming*.

Minimal balanced sets

Definition

Let $Z \subseteq \mathbb{R}_+^N \setminus \{0\}$ be a finite set. We say that Z is *balanced* if there exists a *balancing vector* $(w_z)_{z\in Z'}$ such that $\sum_{z\in Z} w_z z = \mathbb{1}^N$.

- We say that a balanced subset is *minimal* if it does not contain a proper subset that is balanced.

Minimal balanced sets

- Some properties of the minimal balanced collections remain true for the minimal balanced sets.

Lemma

A balanced set is minimal if and only if it has a unique balancing vector.

Proposition

A minimal balanced set contains at most n elements.

Minimal balanced sets

- Consider $z_1, \ldots, z_k$ elements of a set Z with $k \leq n$.
- Define the *weighted incidence matrix* W^Z of a set Z by $W^Z_{i,j} = z^j_i$.

Lemma

Take a finite nonempty set $Z \subseteq \mathbb{R}^N_+$ of k elements and consider its weighted incidence matrix W^Z and its augmented matrix $A^Z = \left[W^Z \mid \mathbb{1}^N \right]$. Z has a unique system of coefficients if and only if $\mathrm{rank}\left(A^Z\right) = \mathrm{rank}\left(W^Z\right) = k$. If all these coefficients are nonnegative, Z' is a minimal balanced subset.

Final algorithm

To check core stability, we have to

1. compute the set of strictly vital-exact coalitions,
2. with these, compute the set of feasible collections,
3. for every feasible collection $\mathcal{S}$, compute $\mathbb{C}(\mathcal{S})$,
4. for every $(\mathcal{B}_S)_{S \in \mathcal{S}} \in \mathbb{C}(\mathcal{S})$, compute the set Z,
5. for every Z, compute the set of its minimal balanced subsets,
6. for every minimal balanced subset, compute the coefficients needed for the weighted sum, and then check the condition of the theorem.

Final algorithm

To improve the efficiency of the algorithm, we can

1. check the balancedness of the game (Bondareva-Shapley),
2. check if there exists a feasible collection $\mathcal{S} = S_1, S_2$ with $S_1 \cup S_2 = N$,
3. check the exactness of the singletons (Gillies[1], 1959),
4. compute the set of extendable coalitions (see Shellshear-Sudhölter[2], 2009)

[1] Gillies, D.B. (1959). Solutions to general non-zero-sum games. *Contributions to the Theory of Games 4*, 47-85.

[2] Shellshear, E., & Sudhölter, P. (2009). On core stability, vital coalitions, and extendability. *Games and Economic Behavior*, 67(2), 633-644.

Final algorithm

Proposition (Gillies)

The core is stable only if the singletons are exact.

Proposition

All the elements of $X_{\mathcal{S}}$ are dominated by a core element if there is a minimal (w.r.t. inclusion) coalition of $\mathcal{S}$ that is extendable.

Examples

Consider the game on $N = \{1, 2, 3\}$ such that:

$$v : \begin{cases} \{i\} & \mapsto & 0 & i \in N, \\ \{i, j\} & \mapsto & 1/2 & i, j \in N, \\ N & \mapsto & 1. & \end{cases}$$

- No proper coalition is effective,
- Feasible collections that do not contain a singleton or an extendable coalition that is minimal: $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$,
- The game is vital-exact extendable: the core is stable,
- CPU time: 0.06 second.

Examples

Consider the game on $N = \{1,2,3,4\}$ such that $v(S) = 0.6$ if $|S| = 3$, $v(N) = 1$ and $v(T) = 0$ otherwise.

- No proper coalition is effective,
- The collection $\{\{1,3,4\},\{1,2,3\}\}$ is feasible, therefore the core cannot be stable,
- CPU time: 0.06 second.

Examples

Let (N, v) the game[1] defined $N = \{1, 2, 3, 4, 5\}$ by
$v(S) = \max\{\lambda_1(S), \lambda_2(S)\}$ with $\lambda_1 = (2, 1, 0, 0, 0)$ and $\lambda_2 = (0, 0, 1, 1, 1)$.

- Effective proper coalitions: $\{2, i\}_{i=3,4,5}$ and $\{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}$,
- Feasible collections that do not contain a singleton or an extendable coalition that is minimal:

$$\left\{ \begin{array}{c} \{\{1,3,4\},\{1,4,5\}\}, \{\{1,3,5\},\{1,4,5\}\}, \{\{1,3,4\},\{1,3,5\}\}, \\ \{\{1,3,4\},\{1,3,5\},\{1,4,5\}\}, \{\{1,3,4\}\}, \{\{1,3,5\}\}, \{\{1,4,5\}\} \end{array} \right\},$$

- The core is not stable because collection $\{\{1, 3, 5\}, \{1, 4, 5\}\}$ does not satisfy the condition of Grabisch and Sudhölter's theorem,
- CPU time: 1.5 second.

[1] Biswas, A. K., et al (1999). Large cores and exactness. *Games and Economic Behavior* 28.1 : 1-12

Examples

We consider the same game as before, but with $v(N) = 3.1$.

- Now, there is no proper coalition that is effective;
- The number of feasible coalitions that does not contain a singleton or an extendable coalition that is minimal increases to 51;
- CPU time: more than 250 hours.

Examples

Let (N, v) be the game[1] defined on $N = \{1, 2, 3, 4, 5, 6\}$ by

$$
\begin{aligned}
v(S) &= 2 \text{ for } S = \begin{cases} \{2,5\}, \{3,5\}, \{1,2,5\}, \{2,3,5\}, \{2,4,5\}, \{2,5,6\}, \{1,2,4,5\} \\ \{1,2,4,6\}, \{1,2,5,6\}, \{2,4,5,6\} \text{ and } \{1,2,4,5,6\}, \end{cases} \\
v(S) &= 3 \text{ for } S = \{3,4,5\}, \\
v(S) &= 4 \text{ for } \ S = \begin{cases} \{3,6\}, \{1,3,5\}, \{1,3,6\}, \{3,4,6\}, \{3,5,6\}, \{1,2,3,5\}, \\ \{1,3,4,5\}, \{1,3,4,6\}, \{1,3,5,6\}, \{2,3,4,5\} \text{ and } \{1,2,3,4,5\}, \end{cases} \\
v(S) &= 6 \text{ for } S = \begin{cases} \{2,3,6\}, \{1,2,3,6\}, \{2,3,4,6\}, \{2,3,5,6\}, \\ \{1,2,3,4,6\} \text{ and } \{1,2,3,5,6\}, \end{cases} \\
v(S) &= 8 \text{ for } S = \{3,4,5,6\}, \{1,3,4,5,6\}, \{2,3,4,5,6\}, \\
v(N) &= 10 \text{ and } v(T) = 0 \text{ otherwise.}
\end{aligned}
$$

- The core is not stable because the collection $\{\{1, 3, 5\}, \{3, 4, 5, 6\}\}$ does not satisfy the condition of Grabisch and Sudhölter's theorem,
- CPU time: 18 minutes and 12 seconds (43 seconds for Peleg's method).

[1] Studený, M., & Kratochvíl, V. (2021). Facets of the cone of exact games.

Thank you for your attention!

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