Decentralized multilateral bargaining

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Outline

1. Nash program
2. Cooperative NTU games
3. Non-cooperative game
4. Conclusions

Section 1

Nash program

Game theory

Cooperative game theory	Non-cooperative game theory

Game theory

Cooperative game theory	Non-cooperative game theory

- Implementation
- Nash program
- Non-cooperative approach

Section 2

Cooperative NTU games

NTU games

Transferable utility (TU) games

- Partial agreements
- Transferable utility
- Shapley value (1953)

NTU games

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- Partial agreements
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Bargaining problems

- Unanimity required
- Non-transferable utility
- Nash solution (1950)

NTU games

Non transferable utility (NTU) games

- Partial agreements
- Non-transferable utility
- Harsanyi value (1963), Shapley NTU value (1969), consistent value (1989, 1992)

Transferable utility (TU) games

- Partial agreements
- Transferable utility
- Shapley value (1953)

Bargaining problems

- Unanimity required
- Non-transferable utility
- Nash solution (1950)

The model

A **Non-Transferable Utility (NTU) game** is a pair (N, V) where:

- $N = \{1, 2, ..., n\}$ is a set of players
- $V: S \subseteq N \longrightarrow V(S) \subset \mathbb{R}^S$ correspondence satisfying:
  - $V(S)$ non-empty, closed, convex, comprehensive, and bounded-above.
  - Superadditivity: $V(S) \times V(T) \subset V(S \cup T)$ for all $S, T \subset N, S \cap T = \varnothing$.
  - $V(S)$ nonlevel: For each x in the frontier of $V(S)$, there exists a unique normalized vector λ orthogonal to $V(S)$ on x with all its coordinates positive.

A **rule** is a function Φ that assigns to each NTU game (N,V) a payoff allocation $\Phi(N,V) \in V(N)$.

Example

Pure exchange economy with three players.

Water grains and water are required to prepare coffee. Sugar is optional.

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$$V(\{1,2\}) = \{x \in \mathbb{R}^{\{1,2\}} : 2x_1 + x_2 \leq 1\}$$
$$V(\{1,3\}) = \{x \in \mathbb{R}^{\{1,3\}} : x_1, x_3 \leq 0\}$$
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$$V(N) = \{x \in \mathbb{R}^{N}: x_1 + x_2 + x_3 \leq 1\}$$

The model

A **Transferable Utility (TU) game** is a pair (N, v) where:

- $N = \{1, 2, ..., n\}$ is a set of players
- $v: S \subseteq N \longrightarrow v(S) \in \mathbb{R}$ correspondence satisfying $v(\varnothing) = 0$.

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Shapley value for TU games:

$Sh_i(N,v) = \sum_{S \subset N: i \in S} d_v(S)/\lvert S \rvert$ where $d_v(S) \in \mathbb{R}$ are the Harsanyi dividends of v.	$Sh_i(N,v) = \sum_{\pi \in \Pi} m_i^{\pi}(v)/\lvert \Pi \rvert$ where $m^{\pi}(v) \in \mathbb{R}^N$ are the marginal contributions vectors of v under order π.

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Any TU game is also an NTU game.

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$$Sh(N,v) = (4,1,1) \qquad Sh(N,V) = (4,1,1)$$

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If the utility is interchangeable at a fixed rate, the game is still (essentially) TU:

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$Sh(N,v) = (4,1,1)$ $\qquad Sh(N,V) = (4,1,1)$ $\qquad Sh(N,V) = (4/\lambda_1, 1/\lambda_2, 1/\lambda_3)$

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4. If $Sh(N,v^\lambda) \in V(N)$, we say that $Sh(N,v^\lambda)$ is a **Shapley NTU** value of (N,V).

The Shapley NTU value (Shapley, 1969)

Pure exchange economy with three players.

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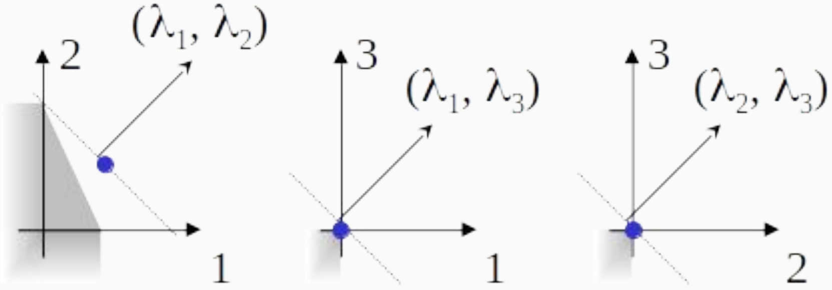

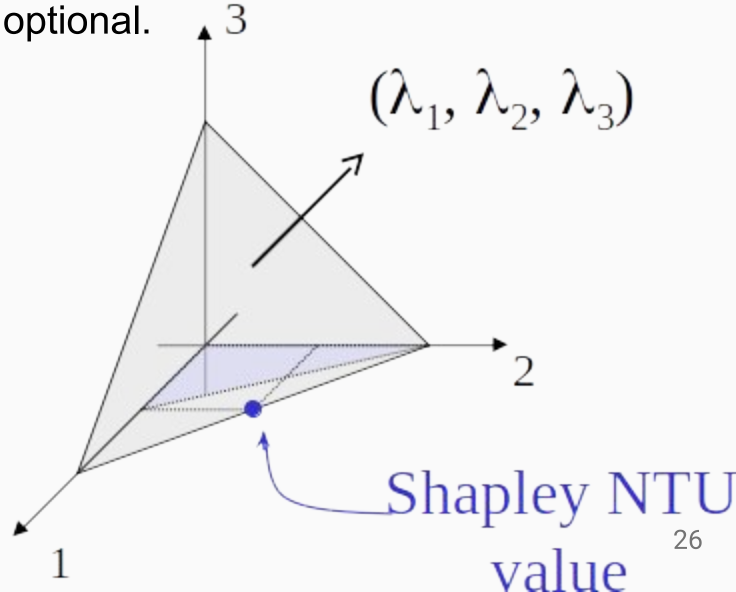

Money as utility (alternative 1)

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The Harsanyi value (Harsanyi, 1963)

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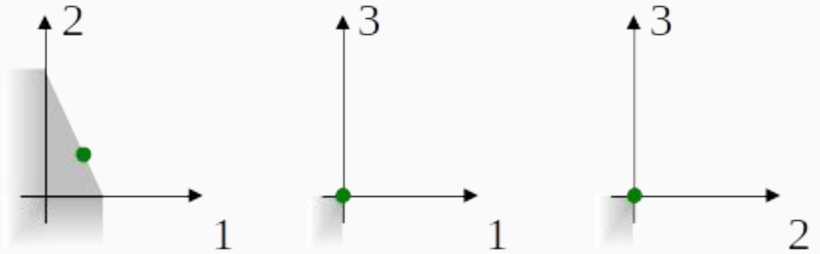

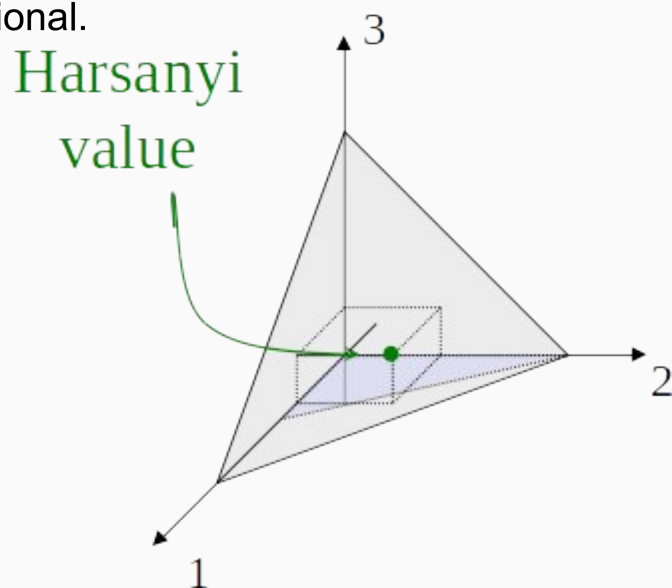

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3. If $C(N,v^\lambda) \in V(N)$, we say that $C(N,v^\lambda)$ is a **consistent value** of (N,V).

The consistent value (Maschler and Owen, 1992)

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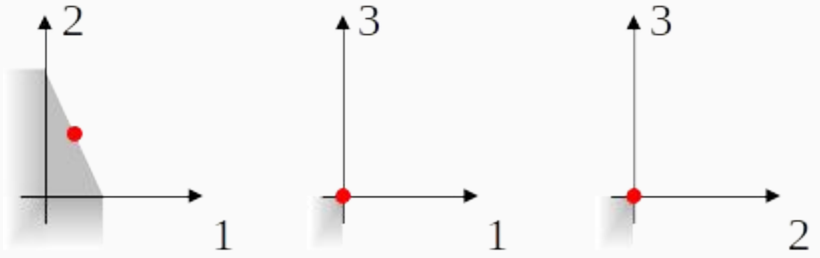

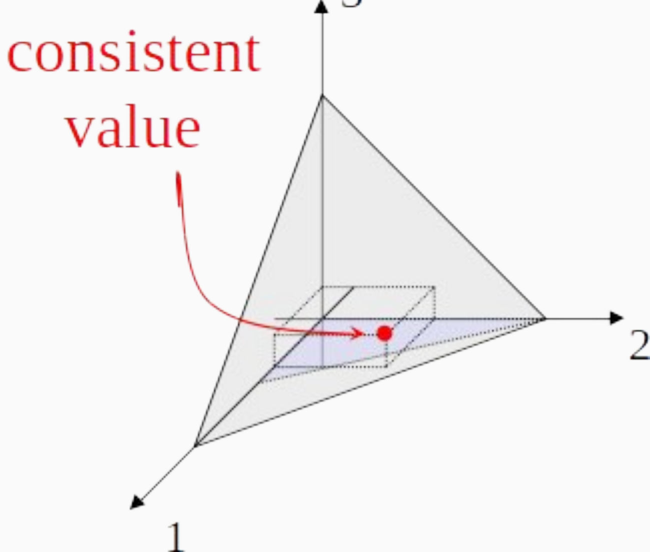

Generalizations of the Shapley value

			Exchange rate	
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procedure	Harsanyi dividends	λ^S		
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Section 3

Non-cooperative game

Implementation of the Nash solution in bargaining games

- Nash (Econometrica, 1953)
- Rubinstein (Econometrica, 1982)
- van Damme (JET, 1986)
- Binmore (“The economics of bargaining”, ed. by Binmore and Dasgupta, 1987)
- Maschler, Owen and Peleg (“The Shapley value”, ed. by Roth, 1988)
- Hart and Mas-Colell (Econometrica, 1996)

Implementation of the Shapley value in TU games

- Gul (Econometrica, 1989)
- Hart and Moore (J Pol Ec, 1990)
- Winter (ET, 1994)
- Evans (GEB, 1992)
- Hart and Mas-Colell (Econometrica, 1996)
- Dasgupta and Chiu (IJGT, 1998)
- Pérez-Castrillo and Wettstein (JET, 2001)
- Vidal-Puga (EJOR, 2008)
- Ju (JME, 2012)

Common features when dealing with partial agreements

- Players "play" (*make offers and counteroffers, agree or disagree, vote, make partial payoffs, ...*) in N.
- Eventually, players split (or some are simply excluded) and the bargaining goes on in some (or several) subcoalition S, without possibility to rejoin.
- The risk of these splits is the tool that make players in N to reach an agreement in equilibrium.

Alternative features when dealing with partial agreements

- Players "play" (make offers and counteroffers, agree or disagree, vote, make partial payoffs, ...) in N, but their offers also consider the payoffs in case of disagreement.
- Players never split (nor are excluded) nor the bargaining goes on in some (or several) subcoalition S.
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Common and alternative features when dealing with partial agreements

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- The risk of disagreement is the tool that make players in N to reach an agreement in equilibrium.

The non-cooperative game: Rounds 1 and 2

An order of the players is randomly chosen (assume $12 \ldots n$).

1. Player 1 presents a rule f: $S \subseteq N \longrightarrow f(S) \subseteq V(S)$.
2. Player 2 either
   a. agrees on f and joins $\{1\}$, or
   b. disagrees and proposes a new rule f^* to player 1.
      i. If player 1 accepts, $\{1,2\}$ forms with rule f^*, and the turn passes to player 3.
      ii. If player 2 rejects, it does not join $\{1\}$ and the turn passes to player 3.

The non-cooperative game: Round r

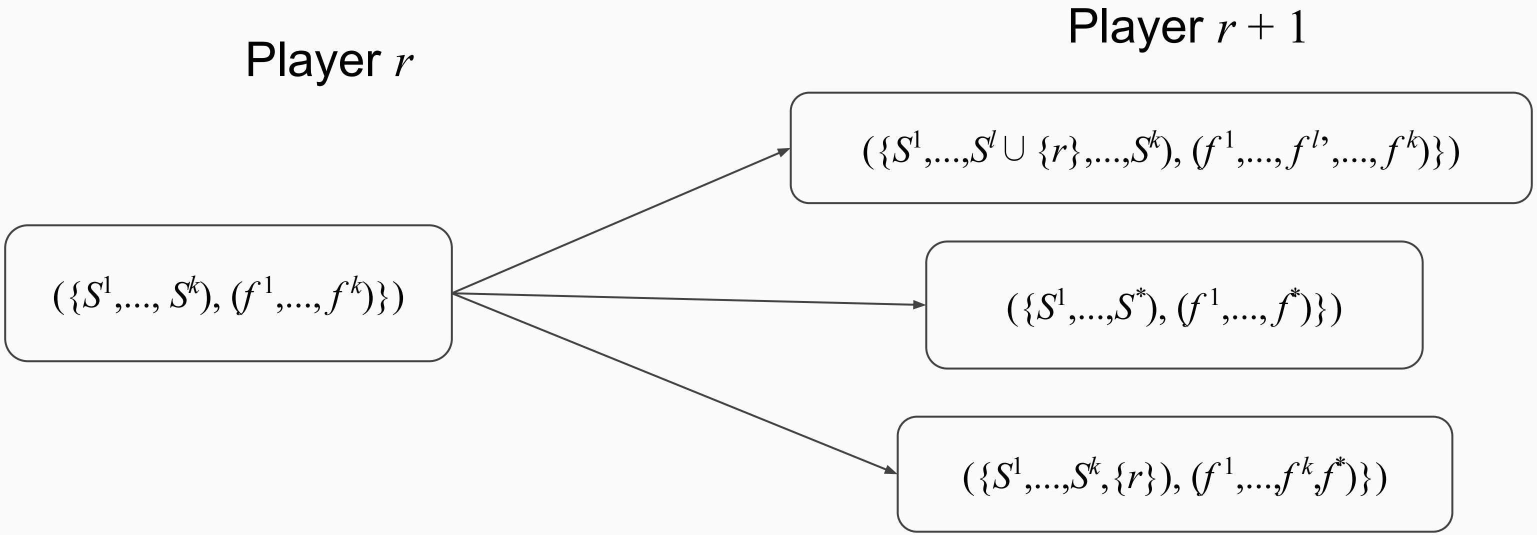
Player r faces $((S^1, f^1),\ldots,(S^k, f^k))$ where

- $\{S^1,\ldots,S^k\}$ is a partition of $\{1,\ldots,r\text{-}1\}$ and
- $(f^1,\ldots,f^k)$ is the vector of rules they have respectively agreed upon.

Player r either

1. agrees on some (S^l, f^l) and joins S^l, or
2. disagrees and proposes a new rule f^* to everyone.
   a. If some coalitions accept (unanimity required inside), they form a new merged coalition with r and rule f^*, and the turn passes to player $r+1$.
   b. If all coalitions reject, player r does not join any coalition and the turn passes to $r+1$ with $((S^1,f^1),\ldots,(S^k,f^k),(\{r\},f^*))$.

Round r


Player r
Player $r + 1$
$(\{S^1,...,S^l \cup \{r\},...,S^k), (f^1,...,f^{l'},...,f^k)\})$
$(\{S^1,..., S^k), (f^1,...,f^k)\})$
$(\{S^1,...,S^*), (f^1,...,f^*)\})$
$(\{S^1,...,S^k,\{r\}), (f^1,...,f^k,f^*)\})$

Last round $(n+1)$

- If we face $((\{N\}),(f))$, i.e., all coalitions have unanimously agreed on a single rule f, then each $i \in N$ receives $f_i(N)$ and the game finishes.
- If we face $((S^1,f^1),\dots,(S^k,f^k))$ with $k > 1$, i.e., there is no unanimity, then
    - With probability $\varrho \in [0,1)$, the whole process is repeated with a new order.
    - With probability $1-\varrho$, each $i \in S^l$ receives $f_i^l(S^l)$ and the game finishes.

Main result

There exists a stationary subgame perfect equilibrium payoff allocation for each order. Moreover, this payoff allocation is efficient and individually rational.

Furthermore, as ϱ approaches 1, the expected final payoff allocation approaches a Shapley NTU value.

Corollary:

- For TU games, the Shapley value is the unique expected equilibrium payoff.
- For bargaining problems, the unique expected equilibrium payoff approaches the Nash bargaining solution as ϱ approaches 1.

Section 4

Conclusions

Summary

Summary:

1. We design a decentralized protocol of bargaining (non-cooperative game) where no players are ever excluded.

2. We determine the final payoffs in equilibrium.

3. The final payoffs approach the Shapley NTU value.

Non-cooperative approaches

- Consistent value: Hart and Mas-Colell (Econometrica, 1996)
- Shapley NTU value: This research.
- Harsanyi value: Open question.