# Axiomatic characterizations of the knapsack and greedy participatory budgeting methods

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# Outline

# Motivation

A participatory budgeting election is a collective decision-making process of direct democracy in which a local government body asks citizens to vote on project proposals to decide how the budgetary spending should be allocated.

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Reasons:

- radical democratic change,
- reduction of the gap between citizens and their elected officials,
- increase of transparency of public finances.

Recent studies estimate that, as of 2018, there have been more than 7000 participatory budgeting initiatives in America, Europe, Africa, and Asia (Cabannes, 2004; Cabannes and Lipietz, 2018; Dias, 2018).

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# Related work

Empirical studies analyze the procedures and results of real-life participatory budgeting elections in a certain town or municipality (Goel et al. 2019; Laruelle, 2021).

Theoretical work originates in the field of computational social choice and broadly investigate the computational, strategic, and fairness properties of participatory budgeting election methods (Talmon and Faliszewski, 2019; Skowron et al., 2020).

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Settings:

- Decision space
  - Discrete: projects must be either fully implemented or not implemented at all.
  - Continuous: projects can be allocated different amounts of funds to implement them to different degrees of effectiveness.
- Ballot design
  - $k$-approval voting: citizens approve the $k$ projects they like the most.
  - Approval voting: citizens approve all projects that they like.
  - Knapsack voting: provide the ideal allocation according to their preferences.

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# Framework

A participatory budgeting instance (briefly, a PB instance) is a tuple $I = (P, c, V, l)$ where

- $P$ is a finite set of project proposals,
- $c : P \to \mathbb{R}_+$ is a cost function assigning a cost $c(p) \geq 0$ to every proposal $p \in P$,
- $V = (V_i)_{i \in N}$ is the voting profile cast by a finite set of voters $N$, where $V_i \subseteq P$ specifies the proposals that voter $i$ approves of,
- $l \geq 0$ is the budget limit.

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Given any PB instance $I = (P, c, V, l)$, the approval score of a project $p \in P$, denoted $a(p)$, is the amount of voters approving $p$, *i.e.*

$$a(p) := |\{i \in N : p \in V_i\}|.$$

We extend $a$ and $c$ from $P$ to $2^P$ by letting

$$c(B) := \sum_{p \in B} c(p) \quad \text{and} \quad a(B) := \sum_{p \in B} a(p), \quad \text{for every } B \subseteq P,$$

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A bundle $B \subseteq P$ is affordable if the total cost of the projects of $B$ respects the budget limit, $i.e$,

$$c(B) \leq l.$$

It is exhaustive if, after implementing $B$, it is not possible to implement any more projects. That is,

$$c(B \cup \{p\}) > l, \text{ for every } p \in P \setminus B.$$

We denote by $\mathcal{AE}(I)$ the set of affordable and exhaustive bundles.

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Given a PB instance $I = (P, c, V, l)$ and a project $q \in P$, the $q$-reduced PB instance is the PB instance obtained from $I$ after implementing and removing $q$ from the set of available proposals. Formally,

$$I^{-q}_{-c(q)} = (P \setminus \{q\}, (V_i \cap (P \setminus \{q\}))_{i \in N}, c|_{P \setminus \{p\}}, l - c(q))).$$

For every affordable bundle $Q \subseteq P$, $I^{-c(Q)}_{-Q}$ is defined similarly.

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# Example

Consider the following PB instance with budget limit $l = 14$:

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ | $\checkmark$ |
| $V_2$ | $\checkmark$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ |
| $V_3$ | $\checkmark$ | $\checkmark$ | $\times$ | $\times$ | $\checkmark$ |
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| $V_3$ | ✓ | ✓ | × | × | ✓ |
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| $V_5$ | × | ✓ | × | × | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

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| $\mathcal{AE}(I)$ | $c(B)$ |
|---|---|
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| $\{p_1, p_2, p_4\}$ | 11 |
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# Framework

A budgeting method, $\mathcal{R}$, is a function associating with every PB instance $I = (P, c, V, l)$ a nonempty set of affordable bundles $\mathcal{R}(I) \subseteq 2^P$.

Every set of projects that is possibly funded according to $\mathcal{R}$, up to a tie-breaking mechanism, $i.e.$ every $B \in \mathcal{R}(I)$, is called a winning bundle.

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# The knapsack method

The knapsack approval method, $\mathcal{K}$, is the budgeting method that selects the set of affordable and exhaustive bundles with the highest approval score.

Formally, for every $I = (P, c, V, l)$,

$$\mathcal{K}(I) := \underset{B \subseteq \mathcal{AE}(I)}{\text{argmax}}\, a(B).$$

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$$\mathcal{K}(I) = \{\{p_1, p_2, p_3\}, \{p_3, p_4, p_5\}\}$$

# The greedy method

The greedy approval method, $\mathcal{G}$, is the budgeting method that sequentially selects the projects with highest approval rate until the budget is exhausted.

# The greedy method

For every PB instance $I = (P, c, V, l)$ and any $\pi \in \Pi(P)$:

- $p_i^{(I,\pi)}$ denotes the project selected at stage $i$;
- $B_i^{(I,\pi)}$ is the bundle of selected projects at stage $i$, with $B_0^{(I,\pi)} := \emptyset$;
- $l_i^{(I,\pi)}$ is the remaining budget after implementing the bundle $B_i^{(I,\pi)}$, with $l_0^{(I,\pi)} := l$;
- $S_i^{(I,\pi)}$ is the set of affordable projects after implementing the bundle $B_i^{(I,\pi)}$, with $S_0^{(I,\pi)} := \{p \in P : c(p) \leq l\}$.

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Assume that stage $k$ is defined, for all $k \leq i-1$.

Stage $i$:

- $p_i^{(I,\pi)} \in S_{i-1}^{(I,\pi)}$ such that, for all $q \in S_{i-1}^{(I,\pi)}$, $q \neq p_i^{(I,\pi)}$,

$$a(p_i^{(I,\pi)}) > a(q) \qquad \text{or} \qquad [a(p_i^{(I,\pi)}) = a(q) \text{ and } \pi(p_i^{(I,\pi)}) < \pi(q)];$$

- $B_i^{(I,\pi)} := B_{i-1}^{(I,\pi)} \cup \{p_i^{(I,\pi)}\}$.
- $l_i^{(I,\pi)} := l_{i-1}^{(I,\pi)} - c(p_i^{(I,\pi)})$;
- $S_i^{(I,\pi)} := \left\{p \in S_{i-1}^{(I,\pi)} : p \notin B_i^{(I,\pi)} \text{ and } c(p) \leq l_i^{(I,\pi)}\right\}$.

This procedure ends when we reach a stage $\delta(I,\pi)$ such that $S_{\delta(I,\pi)}^{(I,\pi)} = \emptyset$.

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  $a(p_i^{(I,\pi)}) > a(q)$ or $[a(p_i^{(I,\pi)}) = a(q)$ and $\pi(p_i^{(I,\pi)}) < \pi(q)]$;

- $B_i^{(I,\pi)} := B_{i-1}^{(I,\pi)} \cup \{p_i^{(I,\pi)}\}$.
- $l_i^{(I,\pi)} := l_{i-1}^{(I,\pi)} - c(p_i^{(I,\pi)})$;
- $S_i^{(I,\pi)} := \left\{p \in S_{i-1}^{(I,\pi)} : p \notin B_i^{(I,\pi)} \text{ and } c(p) \leq l_i^{(I,\pi)}\right\}$.

This procedure ends when we reach a stage $\delta(I,\pi)$ such that $S_{\delta(I,\pi)}^{(I,\pi)} = \emptyset$.

# The greedy method

Assume that stage $k$ is defined, for all $k \leq i-1$.

Stage $i$:

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This procedure ends when we reach a stage $\delta(I,\pi)$ such that $S_{\delta(I,\pi)}^{(I,\pi)} = \emptyset$.

# The greedy method

Then, for every $I = (P, c, V, l)$,

$$\mathcal{G}(I) := \{B \subseteq P : B = B^{(I,\pi)}_{\delta(I,\pi)}, \text{ for some } \pi \in \Pi(P)\}.$$

# The greedy method

Then, for every $I = (P, c, V, l)$,

$$\mathcal{G}(I) := \{B \subseteq P : B = B^{(I,\pi)}_{\delta(I,\pi)},\ \text{for some } \pi \in \Pi(P)\}.$$

# Example

Consider the following PB instance with budget limit $l = 14$:

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | × | ✓ | × | ✓ | ✓ |
| $V_2$ | ✓ | × | ✓ | × | ✓ |
| $V_3$ | ✓ | ✓ | × | × | ✓ |
| $V_4$ | ✓ | × | ✓ | × | ✓ |
| $V_5$ | × | ✓ | × | × | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

- $B_0^{(I,\pi)} = \emptyset$;
- $l_0^{(I,\pi)} = 14$;
- $S_0^{(I,\pi)} = \{p_1, p_2, p_3, p_4, p_5\}$.

# Example

Consider the following PB instance with budget limit $l = 14$:

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ | $\checkmark$ |
| $V_2$ | $\checkmark$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ |
| $V_3$ | $\checkmark$ | $\checkmark$ | $\times$ | $\times$ | $\checkmark$ |
| $V_4$ | $\checkmark$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ |
| $V_5$ | $\times$ | $\checkmark$ | $\times$ | $\times$ | $\checkmark$ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

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|---|---|---|---|---|---|
| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | × | ✓ | × | ✓ | ✓ |
| $V_2$ | ✓ | × | ✓ | × | ✓ |
| $V_3$ | ✓ | ✓ | × | × | ✓ |
| $V_4$ | ✓ | × | ✓ | × | ✓ |
| $V_5$ | × | ✓ | × | × | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

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| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | × | ✓ | × | ✓ | ✓ |
| $V_2$ | ✓ | × | ✓ | × | ✓ |
| $V_3$ | ✓ | ✓ | × | × | ✓ |
| $V_4$ | ✓ | × | ✓ | × | ✓ |
| $V_5$ | × | ✓ | × | × | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

For every $\pi \in \Pi(P)$ such that $\pi(p_1) < \pi(p_2)$:

Stage 1

- $p_1^{(I,\pi)} = p_5$;
- $B_1^{(I,\pi)} = \{p_5\}$;
- $l_1^{(I,\pi)} = 14 - 8 = 6$;
- $S_1^{(I,\pi)} = \{p_1, p_2, p_3, p_4\}$.

# Example

Consider the following PB instance with budget limit $l = 14$:

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | × | ✓ | × | ✓ | ✓ |
| $V_2$ | ✓ | × | ✓ | × | ✓ |
| $V_3$ | ✓ | ✓ | × | × | ✓ |
| $V_4$ | ✓ | × | ✓ | × | ✓ |
| $V_5$ | × | ✓ | × | × | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

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| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | × | ✓ | × | ✓ | ✓ |
| $V_2$ | ✓ | × | ✓ | × | ✓ |
| $V_3$ | ✓ | ✓ | × | × | ✓ |
| $V_4$ | ✓ | × | ✓ | × | ✓ |
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| $a$ | 3 | 3 | 2 | 1 | 5 |

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|---|---|---|---|---|---|
| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ | $\checkmark$ |
| $V_2$ | $\checkmark$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ |
| $V_3$ | $\checkmark$ | $\checkmark$ | $\times$ | $\times$ | $\checkmark$ |
| $V_4$ | $\checkmark$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ |
| $V_5$ | $\times$ | $\checkmark$ | $\times$ | $\times$ | $\checkmark$ |
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| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | $\times$ | ✓ | $\times$ | ✓ | ✓ |
| $V_2$ | ✓ | $\times$ | ✓ | $\times$ | ✓ |
| $V_3$ | ✓ | ✓ | $\times$ | $\times$ | ✓ |
| $V_4$ | ✓ | $\times$ | ✓ | $\times$ | ✓ |
| $V_5$ | $\times$ | ✓ | $\times$ | $\times$ | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

For every $\pi \in \Pi(P)$ such that $\pi(p_1) < \pi(p_2)$:

**Stage 2**

- $p_2^{(I,\pi)} = p_1$;
- $B_2^{(I,\pi)} = \{p_5, p_1\}$;
- $l_2^{(I,\pi)} = 6 - 3 = 3$;
- $S_2^{(I,\pi)} = \{p_4\}$.

# Example

Consider the following PB instance with budget limit $l = 14$:

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | $\times$ | ✓ | $\times$ | ✓ | ✓ |
| $V_2$ | ✓ | $\times$ | ✓ | $\times$ | ✓ |
| $V_3$ | ✓ | ✓ | $\times$ | $\times$ | ✓ |
| $V_4$ | ✓ | $\times$ | ✓ | $\times$ | ✓ |
| $V_5$ | $\times$ | ✓ | $\times$ | $\times$ | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

For every $\pi \in \Pi(P)$ such that $\pi(p_1) < \pi(p_2)$:

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| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | × | ✓ | × | ✓ | ✓ |
| $V_2$ | ✓ | × | ✓ | × | ✓ |
| $V_3$ | ✓ | ✓ | × | × | ✓ |
| $V_4$ | ✓ | × | ✓ | × | ✓ |
| $V_5$ | × | ✓ | × | × | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

For every $\pi \in \Pi(P)$ such that $\pi(p_1) < \pi(p_2)$:

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| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ | $\checkmark$ |
| $V_2$ | $\checkmark$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ |
| $V_3$ | $\checkmark$ | $\checkmark$ | $\times$ | $\times$ | $\checkmark$ |
| $V_4$ | $\checkmark$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ |
| $V_5$ | $\times$ | $\checkmark$ | $\times$ | $\times$ | $\checkmark$ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

For every $\pi \in \Pi(P)$ such that $\pi(p_1) < \pi(p_2)$:

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- $p_2^{(I,\pi)} = p_1$;
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Consider the following PB instance with budget limit $l = 14$:

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
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| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | $\times$ | ✓ | $\times$ | ✓ | ✓ |
| $V_2$ | ✓ | $\times$ | ✓ | $\times$ | ✓ |
| $V_3$ | ✓ | ✓ | $\times$ | $\times$ | ✓ |
| $V_4$ | ✓ | $\times$ | ✓ | $\times$ | ✓ |
| $V_5$ | $\times$ | ✓ | $\times$ | $\times$ | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

For every $\pi \in \Pi(P)$ such that $\pi(p_1) < \pi(p_2)$:

**Stage 3**

- $p_3^{(I,\pi)} = p_4$;
- $B_3^{(I,\pi)} = \{p_5, p_1, p_4\}$;
- $l_3^{(I,\pi)} = 3 - 2 = 1$;
- $S_3^{(I,\pi)} = \emptyset$.

# Example

Consider the following PB instance with budget limit $l = 14$:

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | $\times$ | ✓ | $\times$ | ✓ | ✓ |
| $V_2$ | ✓ | $\times$ | ✓ | $\times$ | ✓ |
| $V_3$ | ✓ | ✓ | $\times$ | $\times$ | ✓ |
| $V_4$ | ✓ | $\times$ | ✓ | $\times$ | ✓ |
| $V_5$ | $\times$ | ✓ | $\times$ | $\times$ | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

For every $\pi \in \Pi(P)$ such that $\pi(p_1) < \pi(p_2)$:

**Stage 3**

- $p_3^{(I,\pi)} = p_4$;
- $B_3^{(I,\pi)} = \{p_5, p_1, p_4\}$;
- $l_3^{(I,\pi)} = 3 - 2 = 1$;
- $S_3^{(I,\pi)} = \emptyset$.

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Consider the following PB instance with budget limit $l = 14$:

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | $\times$ | ✓ | $\times$ | ✓ | ✓ |
| $V_2$ | ✓ | $\times$ | ✓ | $\times$ | ✓ |
| $V_3$ | ✓ | ✓ | $\times$ | $\times$ | ✓ |
| $V_4$ | ✓ | $\times$ | ✓ | $\times$ | ✓ |
| $V_5$ | $\times$ | ✓ | $\times$ | $\times$ | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

For every $\pi \in \Pi(P)$ such that $\pi(p_1) < \pi(p_2)$:

Stage 3

- $p_3^{(I,\pi)} = p_4$;
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Consider the following PB instance with budget limit $l = 14$:

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
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| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | × | ✓ | × | ✓ | ✓ |
| $V_2$ | ✓ | × | ✓ | × | ✓ |
| $V_3$ | ✓ | ✓ | × | × | ✓ |
| $V_4$ | ✓ | × | ✓ | × | ✓ |
| $V_5$ | × | ✓ | × | × | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

For every $\pi \in \Pi(P)$ such that $\pi(p_1) < \pi(p_2)$:

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| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
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| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | × | ✓ | × | ✓ | ✓ |
| $V_2$ | ✓ | × | ✓ | × | ✓ |
| $V_3$ | ✓ | ✓ | × | × | ✓ |
| $V_4$ | ✓ | × | ✓ | × | ✓ |
| $V_5$ | × | ✓ | × | × | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

For every $\pi \in \Pi(P)$ such that $\pi(p_1) < \pi(p_2)$:

The procedure ends at stage $\delta(I, \pi) = 3$.

- $B_3^{(I,\pi)} = \{p_1, p_4, p_5\}$.

# Example

Consider the following PB instance with budget limit $l = 14$:

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ | $\checkmark$ |
| $V_2$ | $\checkmark$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ |
| $V_3$ | $\checkmark$ | $\checkmark$ | $\times$ | $\times$ | $\checkmark$ |
| $V_4$ | $\checkmark$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ |
| $V_5$ | $\times$ | $\checkmark$ | $\times$ | $\times$ | $\checkmark$ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

For every $\pi \in \Pi(P)$ such that $\pi(p_2) < \pi(p_1)$:

### Stage 1

- $p_1^{(I,\pi)} = p_5$;
- $B_1^{(I,\pi)} = \{p_5\}$;
- $l_1^{(I,\pi)} = 14 - 8 = 6$;
- $S_1^{(I,\pi)} = \{p_1, p_2, p_3, p_4\}$.

# Example

Consider the following PB instance with budget limit $l = 14$:

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | $\times$ | ✓ | $\times$ | ✓ | ✓ |
| $V_2$ | ✓ | $\times$ | ✓ | $\times$ | ✓ |
| $V_3$ | ✓ | ✓ | $\times$ | $\times$ | ✓ |
| $V_4$ | ✓ | $\times$ | ✓ | $\times$ | ✓ |
| $V_5$ | $\times$ | ✓ | $\times$ | $\times$ | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

For every $\pi \in \Pi(P)$ such that $\pi(p_2) < \pi(p_1)$:

**Stage 1**

- $p_1^{(I,\pi)} = p_5$;
- $B_1^{(I,\pi)} = \{p_5\}$;
- $l_1^{(I,\pi)} = 14 - 8 = 6$;
- $S_1^{(I,\pi)} = \{p_1, p_2, p_3, p_4\}$.

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Consider the following PB instance with budget limit $l = 14$:

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
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| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | × | ✓ | × | ✓ | ✓ |
| $V_2$ | ✓ | × | ✓ | × | ✓ |
| $V_3$ | ✓ | ✓ | × | × | ✓ |
| $V_4$ | ✓ | × | ✓ | × | ✓ |
| $V_5$ | × | ✓ | × | × | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

For every $\pi \in \Pi(P)$ such that $\pi(p_2) < \pi(p_1)$:

### Stage 1

- $p_1^{(I,\pi)} = p_5$;
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| $V_1$ | × | ✓ | × | ✓ | ✓ |
| $V_2$ | ✓ | × | ✓ | × | ✓ |
| $V_3$ | ✓ | ✓ | × | × | ✓ |
| $V_4$ | ✓ | × | ✓ | × | ✓ |
| $V_5$ | × | ✓ | × | × | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

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| $V_1$ | × | ✓ | × | ✓ | ✓ |
| $V_2$ | ✓ | × | ✓ | × | ✓ |
| $V_3$ | ✓ | ✓ | × | × | ✓ |
| $V_4$ | ✓ | × | ✓ | × | ✓ |
| $V_5$ | × | ✓ | × | × | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

For every $\pi \in \Pi(P)$ such that $\pi(p_2) < \pi(p_1)$:

Stage 2

- $p_2^{(I,\pi)} = p_2$;
- $B_2^{(I,\pi)} = \{p_5, p_2\}$;
- $l_2^{(I,\pi)} = 6 - 6 = 0$;
- $S_2^{(I,\pi)} = \emptyset$.

# Example

Consider the following PB instance with budget limit $l = 14$:

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ | $\checkmark$ |
| $V_2$ | $\checkmark$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ |
| $V_3$ | $\checkmark$ | $\checkmark$ | $\times$ | $\times$ | $\checkmark$ |
| $V_4$ | $\checkmark$ | $\times$ | $\checkmark$ | $\times$ | $\checkmark$ |
| $V_5$ | $\times$ | $\checkmark$ | $\times$ | $\times$ | $\checkmark$ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

For every $\pi \in \Pi(P)$ such that $\pi(p_2) < \pi(p_1)$:

### Stage 2

- $p_2^{(I,\pi)} = p_2$;
- $B_2^{(I,\pi)} = \{p_5, p_2\}$;
- $l_2^{(I,\pi)} = 6 - 6 = 0$;
- $S_2^{(I,\pi)} = \emptyset$.

# Example

Consider the following PB instance with budget limit $l = 14$:

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | × | ✓ | × | ✓ | ✓ |
| $V_2$ | ✓ | × | ✓ | × | ✓ |
| $V_3$ | ✓ | ✓ | × | × | ✓ |
| $V_4$ | ✓ | × | ✓ | × | ✓ |
| $V_5$ | × | ✓ | × | × | ✓ |
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| $V_3$ | ✓ | ✓ | × | × | ✓ |
| $V_4$ | ✓ | × | ✓ | × | ✓ |
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For every $\pi \in \Pi(P)$ such that $\pi(p_2) < \pi(p_1)$:

The procedure ends at stage $\delta(I, \pi) = 2$.

- $B_2^{(I,\pi)} = \{p_2, p_5\}$.

# Example

Consider the following PB instance with budget limit $l = 14$:

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $c$ | 3 | 6 | 4 | 2 | 8 |
| $V_1$ | $\times$ | ✓ | $\times$ | ✓ | ✓ |
| $V_2$ | ✓ | $\times$ | ✓ | $\times$ | ✓ |
| $V_3$ | ✓ | ✓ | $\times$ | $\times$ | ✓ |
| $V_4$ | ✓ | $\times$ | ✓ | $\times$ | ✓ |
| $V_5$ | $\times$ | ✓ | $\times$ | $\times$ | ✓ |
| $a$ | 3 | 3 | 2 | 1 | 5 |

$$\mathcal{G}(I) = \{\{p_1, p_4, p_5\}, \{p_2, p_5\}\}$$

## Properties

**Exhaustiveness**

For every PB instance $I = (P, c, V, l)$, if $B \in \mathcal{R}(I)$, then $B$ is exhaustive.

Interpretation: No additional proposal can be funded with the excess budget remaining after implementing the chosen projects.

This condition, introduced by Talmon and Faliszewski (2019), is based on the intuition that the set of proposals that are submitted to public scrutiny are collectively desirable.

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# Properties

**Positive resposiveness**

For every PB instances $I = (P, c, V, l)$ and $I' = (P, c, V \cup V_j, l)$ where $V_j = B$ for some $j \notin N$,

$$B \in \mathcal{R}(I) \Rightarrow \mathcal{R}(I') = \{B\}.$$

Interpretation: If some bundle of projects is funded, then the addition of a voter who supports exactly those projects that were funded in her absence can only break potential ties in its favor.

# Properties

## Reduction consistency

For every PB instance $I = (P, c, V, l)$, for every $p \in P$ such that $p \in B$ for some $B \in \mathcal{R}(I)$,

$$\{B \in \mathcal{R}(I) : p \in B\} = \left\{\{p\} \cup B' : B' \in \mathcal{R}(I^{-p}_{c(p)})\right\}.$$

Interpretation: Funded bundles remains optimal in the reduced PB instances obtained by implementing chosen projects one-by-one.

# Properties

**Weak reduction consistency**

For every PB instance $I = (P, c, V, l)$, for every $p \in P$ such that $a(p) \geq a(q)$ for every $q \in P$, if $p \in B$ for some $B \in \mathcal{R}(I)$, then

$$\{B \in \mathcal{R}(I) : p \in B\} = \left\{\{p\} \cup B' : B' \in \mathcal{R}(I^{-p}_{c(p)})\right\}.$$

Interpretation: Funded bundles remains optimal in the reduced PB instances obtained by implementing the most approved projects in a winning bundle one-by-one.

# Properties

**K-dominance**

For every PB instance $I = (P, c, V, l)$, for every $B \subseteq P$ and $p \in P$ such that $a(p) \geq a(B)$ and $c(p) \leq l$,

$$B \in \mathcal{R}(I) \Rightarrow p \in B' \text{ for some } B' \in \mathcal{R}(I).$$

Interpretation: If an affordable project is at least as approved as a winning bundle, there exists a winning bundle where this project is funded.

# Properties

**G-dominance**

For every PB instance $I = (P, c, V, l)$, for every $B \subseteq P$ and $p \in P$ such that $a(p) \geq a(q)$ for every $q \in B$ and $c(p) \leq l$,

$$B \in \mathcal{R}(I) \Rightarrow p \in B' \text{ for some } B' \in \mathcal{R}(I).$$

Interpretation: If an affordable project has higher approval rate than any project that is funded, then there must exist another winning bundle in which it is funded.

# Characterizations

**Theorem**

A budgeting method satisfies exhaustiveness, reduction consistency, positive responsiveness, and K-dominance if and only if it is the knapsack approval method.

# Characterizations

**Theorem**

A budgeting method satisfies exhaustiveness, weak reduction consistency, positive responsiveness, and G-dominance if and only if it is the greedy approval method.

## Conclusions

- We axiomatically characterize the knapsack and the greedy approval methods by using comparable sets of axioms.
- As future research, we could adapt our results to different a setting on the voting method.

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**Thank you for your attention!**